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ROGER VILLEMAIRE

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Abstract. While the back-and-forth method has been often attributed to Cantor, it turns out that in the original proof of the characterisation of countable linear dense orders, the mapping is constructed in a single direction. Cameron has called this method Forth and has shown that it can fail to build an automorphism for some homogeneous structures. We give in this paper a characterisation of those homogeneous structures for which Forth always builds an automorphism. This generalises results by Cameron and McLeish.

§1. Introduction. The back-and-forth method is a well-known isomorphism building technique for $\aleph_0$-categorical structures (and more generally for homogeneous structures, see Section 2). The method builds an isomorphism by alternatively picking elements from enumerations of two countable structures. Arguably its most well-known application is the proof of the existence, up to isomorphism, of a unique countable dense total order; namely the set of rational in their natural ordering. But, as Plotkin [14] notes, Cantor's original proof [3] didn't use a back-and-forth construction. Instead, it uses the following method, which Cameron [2] has named *Forth*.

For enumerations $a_1, \dots, a_i, \dots$ and $b_1, \dots, b_i, \dots$ $(i \in \omega)$, Cantor constructs a function f by mapping the i-th element a_i of the first enumeration to a value $f(a_i)$ equal to the first element of the second enumeration, which has the same order relationship with $f(a_1), \dots, f(a_{i-1})$ that a_i has with $a_1, \dots, a_{i-1}$. He then notes the fact that at any step of the construction, $f(a_1), \dots, f(a_n)$ divides the rationals in finitely many intervals and that when $f(a_j)$ is defined the interval containing $f(a_j)$ is split in two, while all other intervals stay unchanged.

Cantor completes the proof by using complete induction to show that f is onto. Namely, he shows that if $\{b_1, \dots, b_n\}$ is in the image $Im(f)$ of f, then this is also the case for b_{n+1}. He proceeds by taking m to be big enough so that $\{f(a_1), \dots, f(a_m)\}$ contains $\{b_1, \dots, b_n\}$. He then notes that if b_{n+1} is not in $\{f(a_1), \dots, f(a_m)\}$, then, by density, there is some a_k, which compares to $\{a_1, \dots, a_m\}$ as b_{n+1} compares to $\{f(a_1), \dots, f(a_m)\}$. The crucial step is now that for the smallest such k, a_k also compares to $\{a_1, \dots, a_{k-1}\}$ as b_{n+1} compares to $\{f(a_1), \dots, f(a_{k-1})\}$. This holds since no a_j $(j = m+1, \dots, k-1)$ compares to $\{a_1, \dots, a_m\}$ as a_k and therefore no such a_j divides the interval containing a_k whose endpoints lay in $\{a_1, \dots, a_m\}$.

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It hence follows, by definition of f, that a_k is mapped to $f(a_k) = b_{n+1}$ and b_{n+1} is indeed in $Im(f)$, completing the proof.

According to Plotkin, who refers to Kueker [12, p. 25] and Hodges [9, p. 130], the back-and-forth method appeared in the mathematical practice in the famous set theory book [7] of Hausdorff. Plotkin also notes that the method was already used in a previous paper of Hausdorff [8] and furthermore in a much less well-known source [10, p.178], which is an expository paper of Huntington.

The question hence remains, as to Cantor's method's strength. Answering a question of Adrian Mathias, Cameron [2, Section 5.2] shows that there are enumerations of $\aleph_0$-categorical structures for which Cantor's construction doesn't lead to an onto map (see also Section 8). Cameron also introduces the following terminology. He says that *Forth suffices* for some structure, if for any pair of enumerations Cantor's method always constructs an onto map. Cameron hence shows that there are some $\aleph_0$-categorical structures for which Forth does not suffice. He furthermore gives a sufficient condition for Forth to suffice.

McLeish [13] generalises Cameron's result and introduces an ordinal rank in term of which he gives a more general sufficient condition for Forth to suffice. Unfortunately he also shows that his condition is still not necessary.

We present in this paper a necessary and sufficient condition for Forth to suffice. In fact, we show in Section 7 that McLeish's rank is given by the greatest fixpoint of a monotone operator. We extend this approach, introducing a construction combining both a least and greatest fixpoint. This allows us to characterise structures for which Forth suffices.

The paper is structured as follows. We recall general facts on homogeneous structures, which are the appropriate setting for the Forth construction, in Section 2. Since we will use them extensively, we give a short presentation of fixpoints and of the closely related Knaster–Tarski construction in Section 3. Our main construction, that of an ordinal rank and of a necessary and sufficient condition for Forth to suffice, is presented in Section 4. That the condition is necessary for Forth to hold is proved in Section 5, while Section 6 shows that it is sufficient. We show in Section 7 that our rank is always smaller that McLeish's, showing in a direct way that our result generalises his. Finally Section 8 presents the computation of our rank on some specific examples.

§2. Homogeneous Structures.

We consider in this paper structures on relational languages, by which we mean a language containing only relation and constant symbols. By a *type* we mean a maximal consistent set of formulas. For $\mathcal{L}$ a relational language, we consider both types in the pure relational language $\mathcal{L}$ and also types in languages $\mathcal{L}_{\bar{a}}$ expanded by a finite tuple of additional constants $\bar{a}$ taken from the universe of some structure. We denote structures using calligraphic script such as $\mathcal{M}$. The universe of the structure $\mathcal{M}$ will be denoted by the corresponding roman script M. As usual, two structures $\mathcal{M}$ and $\mathcal{M}'$ are said to be *elementary equivalent*, in notation $\mathcal{M} \equiv \mathcal{M}'$, if they satisfy exactly the same sentences.

For $\mathcal{M}$ a relational structure, we denote by $\mathcal{G}(\mathcal{M})$ the *group of automorphisms* of $\mathcal{M}$. For $\bar{a} \in M$ a tuple of elements of the universe of $\mathcal{M}$, we denote by $\mathcal{G}(\mathcal{M})_{\bar{a}}$ the subgroup of $\mathcal{G}(\mathcal{M})$ formed of all automorphisms of $\mathcal{M}$ that pointwise fixes $\bar{a}$.

For $\mathcal{M}$ a structure, $\bar{a} \in M$ a tuple of elements of its universe and $\mathcal{G}$ a subgroup of $\mathcal{G}(\mathcal{M})$, the *$\mathcal{G}$-orbit* of $\bar{a}$ is the set $\{g(\bar{a}); g \in \mathcal{G}\}$ of images of $\bar{a}$ under elements of $\mathcal{G}$.

For $\mathcal{M}$ a relational structure in the language $\mathcal{L}$ and $\bar{m} \in M$, the *type* of $\bar{m}$ is the set $tp^{\mathcal{M}}(\bar{m}) = \{\varphi(\bar{x}) \in \mathcal{L}; \mathcal{M} \models \varphi(\bar{m})\}$ of all formulas satisfied by $\bar{m}$. We also denote by $tp^{\mathcal{M}}(-/\bar{m})$ the set of all tuples that satisfy $tp^{\mathcal{M}}(\bar{m})$.

When considering the expanded language $\mathcal{L}_{\bar{a}}$ where $\bar{a} \in M$ are additional constants, the *type* of $\bar{m}$ *over* $\bar{a}$ is the set $tp_{\bar{a}}^{\mathcal{M}}(\bar{m}) = \{\varphi(\bar{x}) \in \mathcal{L}_{\bar{a}}; \mathcal{M} \models \varphi(\bar{m})\}$ of all formulas in the expanded language that are satisfied by $\bar{m}$. As before we denote by $tp_{\bar{a}}^{\mathcal{M}}(-/\bar{m})$ the set of all tuples that satisfy $tp_{\bar{a}}^{\mathcal{M}}(\bar{m})$.

We will write $tp(\bar{m})$ and $tp_{\bar{a}}(\bar{m})$, when $\mathcal{M}$ is clear from the context.

An *enumeration* for a countable set M is a finite or infinite sequence (i.e., indexed by natural numbers) $m_0, m_1, m_2, \ldots$ such that any $m \in M$ is equal to one and only one m_i. While Forth clearly suffices for a finite structure, the definition of our rank makes sense also on finite structures. Moreover, there are finite structures with types of nonzero ranks. We hence always include finite structures in our results, but we will nevertheless somewhat abuse notation and usually write $\langle m_i \in M; i \in \omega\rangle$ for an enumeration, even if for a finite set M, ω should be replaced by one of its initial segments.

The back-and-forth method is the following isomorphism building technique for countable structures. Namely, consider $\mathcal{M}$ and $\mathcal{M}'$ countable structures such that for any pair of tuples $m_1, \ldots, m_n \in M$ and $m'_1, \ldots, m'_n \in M'$ if

$$(\mathcal{M}, m_1, \ldots, m_n) \equiv (\mathcal{M}', m'_1, \ldots, m'_n)$$

both of the following conditions hold:

1. for any $m \in M$, there exists $m' \in M'$ such that $(\mathcal{M}, m_1, \ldots, m_n, m) \equiv (\mathcal{M}, m'_1, \ldots, m'_n, m')$.
2. for any $m' \in M'$, there exists $m \in M$ such that $(\mathcal{M}, m_1, \ldots, m_n, m) \equiv (\mathcal{M}, m'_1, \ldots, m'_n, m')$.

The back-and-forth method builds an isomorphism by alternatively picking an element in M and M', using the above conditions to assure that at each step the sequences of elements chosen from M and M' satisfy the same type. In order to assure that all elements of both M and M' have been picked, the method makes use of enumerations S and S' for M and M' respectively. At each step one hence always chooses the next element of the enumeration, which haven't already been used.

Since the back-and-forth method assures that $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic structures, one can as well consider a single structure. This is usually done with the help of the following definition.

Following [4, p.113] a countable structure $\mathcal{M}$ is said to be *homogeneous* if for any pair of tuples $m_1, \ldots, m_n$ and $m'_1, \ldots, m'_n$ of elements of M such that

$$(\mathcal{M}, m_1, \ldots, m_n) \equiv (\mathcal{M}, m'_1, \ldots, m'_n)$$

and any $m \in M$, there exists $m' \in M$ such that

$$(\mathcal{M}, m_1, \ldots, m_n, m) \equiv (\mathcal{M}, m'_1, \ldots, m'_n, m').$$

For a homogeneous structure $\mathcal{M}$ the back-and-forth method not only constructs an automorphism of $\mathcal{M}$ but it also shows that

$$(\mathcal{M}, m_1, \ldots, m_n) \cong (\mathcal{M}, m'_1, \ldots, m'_n)$$

when

$$(\mathcal{M}, m_1, \ldots, m_n) \equiv (\mathcal{M}, m'_1, \ldots, m'_n)$$

(see [4, p.114]). Two tuples $m_1, \ldots, m_n$ and $m'_1, \ldots, m'_n$ satisfying the same type in $\mathcal{M}$ are therefore on the same $\mathcal{G}(\mathcal{M})$-orbit. Conversely two tuples on the same $\mathcal{G}(\mathcal{M})$-orbit obviously satisfy the same type. This equivalence between types and orbits also extends to types over $\bar{a} \in M$: two tuples $m_1, \ldots, m_n$ and $m'_1, \ldots, m'_n$ satisfy the same type over $\bar{a} \in M$ in $\mathcal{M}$ if and only if they are on the same $\mathcal{G}(\mathcal{M})_{\bar{a}}$-orbit. Homogeneous structures are therefore also characterised as countable structures such that any finite domain partial elementary map (a type preserving partial map) extends to an isomorphism of the complete structure.

Types and orbits are therefore two equivalent approaches to studying homogeneous structures. In this paper we will take the first point-of-view. When using concepts and methods from [2] or [13] we will hence consider the equivalent notions in terms of types.

We will also make use of the following notations. Let f be some partial function on the universe M of a structure $\mathcal{M}$, whose domain contains $\bar{a} = \langle a_1, \ldots, a_n \rangle$. We denote by $f(\bar{a})$ the tuple $\langle f(a_1), \ldots, f(a_n) \rangle$. Furthermore for $p_{\bar{a}}(\bar{x})$ a type in the language $\mathcal{L}_{\bar{a}}$, we denote by $f(p_{\bar{a}}(\bar{x}))$ the *image* of $p_{\bar{a}}(\bar{x})$ under f, which is simply the set of formulas $p_{\bar{a}}(\bar{x})$ in which the elements of $\bar{a}$ have been replaced by those of $f(\bar{a})$. Note that when $\mathcal{M}$ is homogeneous, the image under a partial elementary map of a type realised in $\mathcal{M}$ is also realised in $\mathcal{M}$.

The central concept of this paper is the following. Let $\langle a_i; i \in \omega \rangle$ and $\langle b_i; i \in \omega \rangle$ be two enumerations of some homogeneous structure. Following [2], the *Forth map* for these enumerations is the function which sends a_i to the first b_j in the second enumeration satisfying $f(tp_{a_1,\ldots,a_{i-1}}(a_i))$.

If f is a partial elementary map with domain $\{a_1, \ldots, a_{i-1}\}$, one can apply this construction to extend f to a partial elementary map whose domain is any initial segment of the enumeration $\langle a_i; i \in \omega \rangle$. We will say that these partial maps are *Forth extensions* of f. If we need to explicitly mention the domain, we will say for instance that g is a Forth extension of f *to* $dom(g)$. Furthermore a partial elementary map that has been built using Forth, by this we mean a Forth extensions of the empty map, will be called a *Forth partial map*.

In this paper we will consider, for $\mathcal{M}$ a homogeneous structure, types in the languages $\mathcal{L}_{\bar{a}}$ ($\bar{a} \in M$). We say that a type is *over* $\bar{a}$ if it is a type in the language $\mathcal{L}_{\bar{a}}$. In order to make clear on which language a type is, we will use a subscript as for instance $p_{\bar{a}}$ for a type in $\mathcal{L}_{\bar{a}}$. By a type we hence always mean a type over a *finite* number of constants from M, which is furthermore realised in $\mathcal{M}$.

In order to present our results, we will need to extend the usual terminology slightly. For instance, we will say that a single element m *realises* a type $p_{\bar{a}}(\bar{x})$, even when $\bar{x}$ is of length greater than 1. In that case we just mean that there is some $\bar{m}$ of the right length, containing m and satisfying $p_{\bar{a}}(\bar{x})$ in the usual sense. Note that in a homogeneous structure that realises $p_{\bar{a}}(\bar{x})$, m realises $p_{\bar{a}}(\bar{x})$ if and only if m realises all formulas of $p_{\bar{a}}(\bar{x})$ in some single variable $x \in \bar{x}$.

We will also say that a type $p_{\bar{a}}(x)$, in a single variable, is *realised* in some tuple $\bar{m}$ if there is an $m \in \bar{m}$, which realises $p_{\bar{a}}(x)$ in the usual sense.

Let $\mathcal{M}$ be a homogeneous structure. The group of automorphisms of $\mathcal{M}$ acts naturally on the set of types, sending the type $tp_{\bar{a}}(\bar{m})$ to the type $f(tp_{\bar{a}}(\bar{m}))$, which is the type $tp_{f(\bar{a})}(f(\bar{m}))$ of $f(\bar{m})$ over $f(\bar{a})$. We therefore have the following result, where $M\backslash\bar{a}$ is the set of all elements of M that are not among $\bar{a}$.

PROPOSITION 2.1. *The image $f(tp_{\bar{a}}(\bar{x}))$ of a type $tp_{\bar{a}}(\bar{x})$ realised in $M\backslash\bar{a}$, is a type realised in $M\backslash f(\bar{a})$.*

Finally, for S, S' sets and $\bar{s} = (s_1, \dots, s_n)$ a tuple, we will not only write, as usual, $\bar{s} \in S$ when $s_i \in S$ for all $i = 1, \dots, n$, but also $\bar{s} \in S\backslash S'$, when $s_i \in S$ and $s_i \notin S'$, for all $i = 1, \dots, n$.

§3. Fixpoints.

Since our construction uses least and greatest fixpoints of monotone operators, we rapidly recall in this section the Knaster–Tarski Theorem [11,15]. We will present in particular the fixpoint constructions by ordinal induction, such as in [5].

While the results hold generally for complete lattices, we will restrict the presentation to the lattice of subsets of some set. This is sufficient for the applications we have in mind.

Consider some set S. By an *operator* on S we mean a function $\mathcal{O} : \mathcal{P}(S) \to \mathcal{P}(S)$, where $\mathcal{P}(S)$ is the power set of S (the set of its subsets). An operator $\mathcal{O}$ is said to be *monotone* if $\mathcal{O}(X) \subseteq \mathcal{O}(Y)$, when $X \subseteq Y$. A *fixpoint* of $\mathcal{O}$ is a subset X of S such that $\mathcal{O}(X) = X$.

The Knaster–Tarski fixpoint Theorem states that a monotone operator always has a fixpoint. Furthermore, it has both a *least* and a *greatest* fixpoint (under inclusion). Finally the least and greatest fixpoints can be computed by ordinal induction as follows.

The least fixpoint of a monotone operator $\mathcal{O}$ can be computed using the following inductive definition.

$$\begin{aligned} \mathcal{LO}(0) &= \varnothing, \\ \mathcal{LO}(\alpha + 1) &= \mathcal{O}(\mathcal{LO}(\alpha)), \\ \mathcal{LO}(\alpha) &= \bigcup_{\beta<\alpha} \mathcal{LO}(\beta) \qquad \text{for a limit ordinal } \alpha. \end{aligned}$$

The following result can be proved by ordinal induction.

PROPOSITION 3.1. *The sequence $\mathcal{LO}(\alpha)$ is increasing, i.e., $\mathcal{LO}(\beta) \subseteq \mathcal{LO}(\alpha)$ for ordinals $\beta < \alpha$.*

By picking, when possible, an element that is in $\mathcal{LO}(\alpha + 1)$ while not being in $\mathcal{LO}(\alpha)$, one can show the following.

PROPOSITION 3.2. *The sequence $\mathcal{LO}(\alpha)$ stabilises at an ordinal of cardinality at most $|S|$, i.e., there is an ordinal α of at most the cardinality of S, such that $\mathcal{LO}(\alpha) = \mathcal{LO}(\beta)$ for all $\beta > \alpha$. Furthermore this $\mathcal{LO}(\alpha)$ is the least fixpoint of the operator $\mathcal{O}$.*

For the greatest fixpoint, one can dually consider the following inductive definition.

$$\mathcal{GO}(0) = S,$$
$$\mathcal{GO}(\alpha+1) = \mathcal{O}(\mathcal{GO}(\alpha)),$$
$$\mathcal{GO}(\alpha) = \bigcap_{\beta<\alpha} \mathcal{GO}(\beta) \qquad \text{for a limit ordinal } \alpha.$$

As before we have the following results.

PROPOSITION 3.3. *The sequence $\mathcal{GO}(\alpha)$ is decreasing, i.e., $\mathcal{GO}(\beta) \supseteq \mathcal{GO}(\alpha)$ for ordinals $\beta < \alpha$.*

PROPOSITION 3.4. *The sequence $\mathcal{GO}(\alpha)$ stabilises at an ordinal at most $|S|$, i.e., there is an ordinal α at most the cardinality of S, such that $\mathcal{GO}(\alpha) = \mathcal{GO}(\beta)$ for all $\beta > \alpha$. Furthermore this $\mathcal{GO}(\alpha)$ is the greatest fixpoint of the operator $\mathcal{O}$.*

We will write $\mathcal{GO}(\infty)$ for the greatest fixpoint of operator $\mathcal{O}$.

Both fixpoints computations allow to define an ordinal rank function on S. We will present this result only in the greatest fixpoint case, since this is all we will use.

Namely, from the above greatest fixpoint computation one can define a rank function rk in the following way. For $s \in S$, let $rk(s)$ be the (unique) ordinal α (if its exists) such that $s \in \mathcal{GO}(\alpha)\backslash\mathcal{GO}(\alpha+1)$. If this ordinal $rk(s)$ does exist, it follows that $s \notin \mathcal{GO}_{\mathcal{M}}(\infty)$. We will say in this case that s is of *ordinal rank*. Conversely if there is no such ordinal, then s is in $\mathcal{GO}(\alpha)$ for all ordinals α and hence $s \in \mathcal{GO}(\infty)$. In that case, we will say that s is of *infinite rank*.

REMARK 3.5. Note that, since $\mathcal{GO}(\alpha)$ is a decreasing sequence, $rk(s) \geqslant \alpha$ holds if and only if $s \in \mathcal{GO}(\alpha)$.

In the next section, we will consider an operator on a countable set. We will hence obtain a countable ordinal rank.

§4. Outer-extensions and hereditary expansions.

Let $\mathcal{M}$ be a homogeneous structure. We will denote by $\mathcal{P}_{\mathcal{M}}$ the set of all types $tp_{\bar{a}}(\bar{m})$, such that $\bar{a}$, $\bar{m}$ are tuples of elements of M *with no common element*. For now on, when we speak of a *type*, we always mean an element of $\mathcal{P}_{\mathcal{M}}$ for some homogeneous structure $\mathcal{M}$. Note that $\mathcal{P}_{\mathcal{M}}$ is a countable set.

Let $\mathcal{M}$ be a homogeneous structure and $\bar{m}, \bar{a} \in M$. Inspired by Cameron's concept of type splitting [2, Section 5.2] (see also [13]), we say that $b \in M\backslash\bar{a}$ *outer-extends* $tp_{\bar{a}}(\bar{m})$ if b does not satisfy $tp_{\bar{a}}(\bar{m})$. In such a case we will also say that $tp_{\bar{a},b}(\bar{m})$ is an *elementary outer-extension* (EOE) of $tp_{\bar{a}}(\bar{m})$. An elementary outer-extension is hence a more specific type where constraints relative to an (single) additional element b, which doesn't satisfy the original type, are now considered.

Note that by definition an EOE of a type of $\mathcal{P}_{\mathcal{M}}$ is always in $\mathcal{P}_{\mathcal{M}}$, namely constants and parameters don't have any common element. This is consistent with the fact that we are considering only types of $\mathcal{P}_{\mathcal{M}}$.

We define the *outer-closure* of $tp_{\bar{a}}(\bar{m})$ to be the smallest set of types containing $tp_{\bar{a}}(\bar{m})$ and closed under adding elementary outer-extensions. Namely $tp_{\bar{a},b_1,\dots,b_n}(\bar{m})$ is in the outer-closure of the type $tp_{\bar{a}}(\bar{m})$ if for all $i = 1,\dots,n$, $tp_{\bar{a},b_1,\dots,b_i}(\bar{m})$ is an elementary outer extension of $tp_{\bar{a},b_1,\dots,b_{i-1}}(\bar{m})$. We denote this set by $OC(tp_{\bar{a}}(\bar{m}))$

and say that its elements are *outer-extensions* of $tp_{\bar{a}}(\bar{m})$. Note that the outer-closure $OC(p)$ is characterised as the smallest fixpoint of the following monotone operator:

$$\mathcal{OC}_p(X) = \{q; q = p \text{ or } q \text{ is an EOE of an element of } X\}.$$

For $b \in M$, we will also say that a type $tp_{\bar{a}}(\bar{m})$ *outer-extends beyond* b if it has an outer-extension $tp_{\bar{a},\bar{b}}(\bar{m})$ with $b \in \bar{b}$. In that case we also say that $tp_{\bar{a}}(\bar{m})$ *outer-extends beyond* b to $tp_{\bar{a},\bar{b}}(\bar{m})$. Note that in these cases we can as well consider that b is the last element of $\bar{b}$.

In a similar way, for $b \in M$, we say that a type $tp_{\bar{a}}(\bar{m})$ *outer-extends beyond b's type* if it outer-extends beyond b', for some b' realising $tp_{\bar{a}}(b)$. Again, when $tp_{\bar{a},\bar{b}'}(\bar{m})$ is such an outer-extension ($b' \in \bar{b}'$), we say that $tp_{\bar{a}}(\bar{m})$ *outer-extends beyond b's type* to $tp_{\bar{a},\bar{b}'}(\bar{m})$.

We can now introduce the central concept of this paper. Let T be a subset of $\mathcal{P}_{\mathcal{M}}$. We will say that $tp_{\bar{a}}(\bar{m})$ is T*-expandable* if for all $b \in M \backslash \bar{a}$ either $tp_{\bar{a}}(\bar{m})$ outer-extends beyond b to a type in T or $tp_{\bar{a}}(\bar{m}, b)$ outer-extends beyond b's type to a type in T.

Note that in the previous definition the presence of duplicate elements in $\bar{m}$ are irrelevant since an element satisfies the type of a tuple with duplicates if and only if it satisfies the type of the same tuple without duplicates.

Finally we define the set of *hereditary expandable* types to be the greatest fixpoint of the following operator, which is clearly monotone.

$$\mathcal{HE}(T) = \{p \in T; p \text{ is } T\text{-expandable}\}.$$

As shown in Section 3, $\mathcal{GHE}(\alpha)$ (that we will shorten to $\mathcal{HE}(\alpha)$) allows to define a rank on the set of types $\mathcal{P}_{\mathcal{M}}$. We will denote this rank simply by rk. Note that since $\mathcal{P}_{\mathcal{M}}$ is a countable set, this rank, when not infinite, is a countable ordinal.

Our main result is the following.

THEOREM 4.1. *Forth suffices for some homogeneous structure $\mathcal{M}$ if and only if there is no hereditary expandable type over $\mathcal{M}$ (equivalently if $\mathcal{HE}(\infty) = \emptyset$ or if every type has an ordinal rank).*

Before we consider the proof of this theorem, some remarks are in order.

We consider a tuple $\bar{a}$ as an ordered sequence. While this order is irrelevant when considering a type $tp_{\bar{a}}(\bar{m})$, the order on elements $\bar{b}$ does make a difference when stating that $tp_{\bar{a},\bar{b}}(\bar{m})$ is an outer-extension of $tp_{\bar{a}}(\bar{m})$.

As for T-expandable types, let us recall that by definition, a type p is in $\mathcal{HE}(\alpha+1)$ exactly when p is $\mathcal{HE}(\alpha)$-expandable. For a limit ordinal γ, a type p is in $\mathcal{HE}(\gamma)$ exactly when p is $\mathcal{HE}(\beta)$-expandable, for all $\beta < \gamma$.

As we said in Section 3, a type p is of rank α, if $p \in \mathcal{HE}(\alpha) \backslash \mathcal{HE}(\alpha + 1)$. It then follows that such a p is not in $\mathcal{HE}(\alpha + 1)$, hence p is not $\mathcal{HE}(\alpha)$-expandable. By definition this means that there is some $b \in M \backslash \bar{a}$ such that, if $p = tp_{\bar{a}}(\bar{m})$, then neither $tp_{\bar{a}}(\bar{m})$ outer-extends beyond b to a type in $\mathcal{HE}(\alpha)$ nor $tp_{\bar{a}}(\bar{m}, b)$ outer-extends beyond b's type to a type in $\mathcal{HE}(\alpha)$.

An aspect of the definition of T-expandable types that could seem surprising at first glance is that the b in its definition could be an element of $\bar{m}$. More precisely if a type $tp_{\bar{a}}(\bar{m})$ is T-expandable, then we have in particular that for $m \in \bar{m}$ either $tp_{\bar{a}}(\bar{m})$ outer-extends beyond m to a type in T or $tp_{\bar{a}}(\bar{m}, m)$ outer-extends beyond

m's type to a type in T. Since the first alternative is clearly impossible, in order to be T-expandable, a type must satisfy the second. Furthermore, as we said before, duplicate elements of $\bar{m}$ are irrelevant, hence this second case is equivalent to $tp_{\bar{a}}(\bar{m})$ outer-extending beyond m's type to a type in T. With this simple remark, we can easily show the following result.

PROPOSITION 4.2. *Let $\mathcal{M}$ be a homogeneous structure and $\bar{m}, \bar{a} \in M$. If the set $tp_{\bar{a}}(-/m)$ is finite for some $m \in \bar{m}$, then $rk(tp_{\bar{a}}(\bar{m})) \leqslant |tp_{\bar{a}}(-/m)| - 1$.*

PROOF. We show the result by induction on the cardinality of $tp_{\bar{a}}(-/m)$.

Consider a $tp_{\bar{a}}(\bar{m})$ such that $|tp_{\bar{a}}(-/m)| = 1$. It follows that $tp_{\bar{a}}(\bar{m})$ cannot extend beyond $tp_{\bar{a}}(m)$, since m is the unique element satisfying this type. We hence have, from the previous remark, that $rk(tp_{\bar{a}}(\bar{m})) = 0$ as claimed.

Suppose by induction that the result holds for all m such that $|tp_{\bar{a}}(-/m)| < n$. Consider now $tp_{\bar{a}}(\bar{m})$ to be some type such that $|tp_{\bar{a}}(-/m)| = n$. We have to show that $rk(tp_{\bar{a}}(\bar{m})) \leqslant n - 1$.

If $tp_{\bar{a}}(\bar{m})$ does not extend beyond m's type, then as before its rank is 0 and the claim is fulfilled. Otherwise $tp_{\bar{a}}(\bar{m})$ extends beyond m's type to some $tp_{\bar{a},\bar{b},m'}(\bar{m})$, where m' satisfies $tp_{\bar{a}}(m)$. Since m' satisfies $tp_{\bar{a}}(m)$ but not $tp_{\bar{a},\bar{b}}(\bar{m})$, it follows that $tp_{\bar{a},\bar{b},m'}(-/m)$ is of cardinality strictly lower than n. By induction hypothesis, its rank is strictly smaller than $n - 1$. This means that $tp_{\bar{a},\bar{b},m'}(-/m)$ is not $\mathcal{HE}$(n-1)-expandable.

As $tp_{\bar{a}}(\bar{m})$ does not extend beyond m's type to some $\mathcal{HE}(n-1)$-expandable type, it follows from the previous remark that the rank of $tp_{\bar{a}}(\bar{m})$ is at most $n - 1$, i.e., $rk(tp_{\bar{a}}(\bar{m})) \leqslant n - 1$, which completes the proof. ⊣

Using the action of the automorphism group on types, one shows the following result.

PROPOSITION 4.3. *Let $\mathcal{M}$ be a homogeneous structure. If $tp(m, \bar{a}) = tp(m', \bar{a}')$ then $rk(tp_{\bar{a}}(m)) = rk(tp_{\bar{a}'}(m'))$.*

PROOF. This follows from the fact that there is an automorphism of $\mathcal{M}$ sending $\langle m, \bar{a}\rangle$ to $\langle m', \bar{a}'\rangle$ and that the action of the group of automorphisms of $\mathcal{M}$ on types preserves ranks. ⊣

We will make use of the following corollary.

COROLLARY 4.4. *A partial elementary map f induces a rank preserving map from types over $dom(f)$ (the domain of f) to types over $codom(f)$ (the codomain of f).*

§5. Infinite rank: Forth does not suffice. We will prove in this section the first half of our main result, namely that if a homogeneous structure has a type of infinite rank, then Forth does not suffice.

We will first need some elementary facts about outer-extensions and our rank function.

PROPOSITION 5.1. *If $tp_{\bar{a},\bar{b}}(\bar{m}, \bar{m}')$ is an outer-extension of the type $tp_{\bar{a}}(\bar{m}, \bar{m}')$, then $tp_{\bar{a},\bar{b}}(\bar{m})$ is an outer-extension of $tp_{\bar{a}}(\bar{m})$.*

PROOF. It is sufficient to show that the result holds when $\bar{b}$ is a single element b. In that case $tp_{\bar{a},b}(\bar{m}, \bar{m}')$ is an elementary outer-extension of $tp_{\bar{a}}(\bar{m}, \bar{m}')$ and we have to show that $tp_{\bar{a},b}(\bar{m})$ is an elementary outer-extension of $tp_{\bar{a}}(\bar{m})$. It hence is sufficient to show that b doesn't satisfy $tp_{\bar{a}}(\bar{m})$, when it does not satisfy $tp_{\bar{a}}(\bar{m}, \bar{m}')$.

But this follows from the fact that if b is in some $\bar{b}'$ satisfying $tp_{\bar{a}}(\bar{m})$, by homogeneity $\bar{b}'$ can be extended to some $\langle \bar{b}', \bar{b}'' \rangle$ satisfying $tp_{\bar{a}}(\bar{m}, \bar{m}')$. ⊣

A useful corollary is the following.

Corollary 5.2. *A type $tp_{\bar{a}}(\bar{m})$ that doesn't outer-extend beyond the type of some element $b \in M \backslash \bar{a}$ is of rank* 0.

Proof. In order to show that $tp_{\bar{a}}(\bar{m})$ is of rank 0, we have to prove that $tp_{\bar{a}}(\bar{m}) \notin \mathcal{HE}(1)$, hence that $tp_{\bar{a}}(\bar{m})$ is not $\mathcal{HE}(0)$-expandable. As $\mathcal{HE}(0) = \mathcal{P}_{\mathcal{M}}$, we therefore must show that $tp_{\bar{a}}(\bar{m})$ is not $\mathcal{P}_{\mathcal{M}}$-expandable. By definition this means that we must find some $b \in M \backslash \bar{a}$ such that neither $tp_{\bar{a}}(\bar{m})$ outer-extends beyond b nor $tp_{\bar{a}}(\bar{m}, b)$ outer-extends beyond b's type.

Take b to be as in the statement of this corollary. Since b obviously satisfies its own type, we have that $tp_{\bar{a}}(\bar{m})$ doesn't outer-extend beyond b. It remains to be shown that $tp_{\bar{a}}(\bar{m}, b)$ doesn't outer-extend beyond b's type. But if $tp_{\bar{a}}(\bar{m}, b)$ did outer-extend beyond b's type, then by the previous proposition, $tp_{\bar{a}}(\bar{m})$ would also outer-extend beyond b's type, which is impossible by hypothesis. ⊣

The previous proposition also yields the following rank inequality.

Proposition 5.3. $rk(tp_{\bar{a}}(\bar{m}, \bar{m}')) \leqslant rk(tp_{\bar{a}}(\bar{m}))$.

Proof. It is sufficient to show that $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\alpha)$ when $tp_{\bar{a}}(\bar{m}, \bar{m}') \in \mathcal{HE}(\alpha)$. We will show this by ordinal induction on α. The cases when α is equal to 0 or a limit ordinal being clear, we will now consider the successor ordinal case.

Take $\alpha = \gamma + 1$, a successor ordinal. If $tp_{\bar{a}}(\bar{m}, \bar{m}') \in \mathcal{HE}(\gamma + 1)$, then for any $b \in M \backslash \bar{a}$ either $tp_{\bar{a}}(\bar{m}, \bar{m}')$ outer-extends beyond b to a type in $\mathcal{HE}(\gamma)$ or $tp_{\bar{a}}(\bar{m}, \bar{m}', b)$ outer-extends beyond b's type to a type in $\mathcal{HE}(\gamma)$. Using Proposition 5.1 and the induction hypothesis, we have that either $tp_{\bar{a}}(\bar{m})$ outer-extends beyond b to a type in $\mathcal{HE}(\gamma)$ or $tp_{\bar{a}}(\bar{m}, b)$ outer-extends beyond b's type to a type in $\mathcal{HE}(\gamma)$. We therefore have that $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\alpha)$ as required. ⊣

One can furthermore show the following rank inequality for outer-extensions.

Proposition 5.4. *If $tp_{\bar{a},\bar{b}}(\bar{m})$ is an outer-extension of the type $tp_{\bar{a}}(\bar{m})$, then $rk(tp_{\bar{a},\bar{b}}(\bar{m})) \leqslant rk(tp_{\bar{a}}(\bar{m}))$.*

Proof. As in the proof of the previous proposition, it is sufficient to show that $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\alpha)$ when $tp_{\bar{a},\bar{b}}(\bar{m}) \in \mathcal{HE}(\alpha)$. We will show this by ordinal induction on α. As before, the cases when α is equal to 0 or a limit ordinal are clear.

Consider now $\alpha = \gamma + 1$ a successor ordinal and some $tp_{\bar{a},\bar{b}}(\bar{m}) \in \mathcal{HE}(\alpha)$. In order to show that $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\alpha)$, we have to show that: either $tp_{\bar{a}}(\bar{m})$ outer-extends beyond b to a type in $\mathcal{HE}(\gamma)$ or $tp_{\bar{a}}(\bar{m}, b)$ outer-extends beyond b's type to a type in $\mathcal{HE}(\gamma)$, and this for all $b \in M \backslash \bar{a}$. Now since $tp_{\bar{a},\bar{b}}(\bar{m})$ is in $\mathcal{HE}(\alpha)$, it is also in $\mathcal{HE}(\gamma)$. This settles the case of all b's such that $b \in \bar{b}$. We still have to show the property for $b \in M \backslash (\bar{a}, \bar{b})$.

Consider therefore some $b \in M \backslash (\bar{a}, \bar{b})$. Since $tp_{\bar{a},\bar{b}}(\bar{m}) \in \mathcal{HE}(\gamma + 1)$, it follows that either $tp_{\bar{a},\bar{b}}(\bar{m})$ outer-extends beyond b to a type in $\mathcal{HE}(\gamma)$ or $tp_{\bar{a},\bar{b}}(\bar{m}, b)$ outer-extends beyond b's type to a type in $\mathcal{HE}(\gamma)$.

Now since, by assumption, an outer-extension of $tp_{\bar{a},\bar{b}}(\bar{m})$ is also an outer extension of $tp_{\bar{a}}(\bar{m})$, we have that $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\alpha)$ as required. ⊣

Corollary 5.2 can be generalised to higher ranks in the following way.

Proposition 5.5. *Let $\mathcal{M}$ be a homogeneous structure and $tp_{\bar{a}}(\bar{m})$ some type in $\mathcal{HE}(\alpha+1)$. For any $b \in M\backslash\bar{a}$, there exists $\bar{b}' \in M\backslash\bar{a}$ such that*

1. *$\bar{b}'$ realises $tp_{\bar{a}}(b)$,*
2. *$tp_{\bar{a},\bar{b}'}(\bar{m}) \in \mathcal{HE}(\alpha)$ and is an outer-extension of $tp_{\bar{a}}(\bar{m})$.*

Proof. The type $tp_{\bar{a}}(\bar{m})$ is in $\mathcal{HE}(\alpha+1) = \mathcal{HE}(\mathcal{HE}(\alpha))$. It follows that $tp_{\bar{a}}(\bar{m})$ is $\mathcal{HE}(\alpha)$-*expandable*. We therefore have that either $tp_{\bar{a}}(\bar{m})$ outer-extends beyond b to a type in $\mathcal{HE}(\alpha)$ or $tp_{\bar{a}}(\bar{m}, b)$ outer-extends beyond b's type to a type in $\mathcal{HE}(\alpha)$.

In the first case the claim obviously holds. In the second case there is some $\bar{b}' \in M\backslash\bar{a}$ realising $tp_{\bar{a}}(b)$ and such that $tp_{\bar{a},\bar{b}'}(\bar{m}, b) \in \mathcal{HE}(\alpha)$ and is an outer-extension of $tp_{\bar{a}}(\bar{m}, b)$. Now by Proposition 5.1, $tp_{\bar{a},\bar{b}'}(\bar{m})$ is an outer-extension of $tp_{\bar{a}}(\bar{m})$ and by Proposition 5.3, $rk(tp_{\bar{a},\bar{b}'}(\bar{m})) \geqslant rk(tp_{\bar{a},\bar{b}'}(\bar{m}, b)) \geqslant \alpha$. It hence follows from Remark 3.5 that $tp_{\bar{a},\bar{b}'}(\bar{m}) \in \mathcal{HE}(\alpha)$, completing the proof. ⊣

In order to construct an enumeration for which Forth does not suffice, we will make use of the following result.

Proposition 5.6. *Let $\mathcal{M}$ be a homogeneous structure. Take $\bar{s}_1$, $\bar{s}_2$ finite sequences of elements of M. Let f be a partial elementary map with domain $\bar{s}_1$ and codomain $\bar{s}_2$ and let $\bar{m}$ be formed of the elements of $\bar{s}_2\backslash Im(f)$ (in some order). For $tp_{f(\bar{s}_1),\bar{b}}(\bar{m})$ an outer-extension of $tp_{f(\bar{s}_1)}(\bar{m})$ there exists $\bar{a}$ disjoint from $\bar{s}_1$, such that the forth extension of f to a mapping of $\langle\bar{s}_1, \bar{a}\rangle$ to $\langle\bar{s}_2, \bar{b}\rangle$ maps $\bar{a}$ to $\bar{b}$.*

Proof. It is sufficient to show the result when the sequence $\bar{b}$ is a single element b and $tp_{f(\bar{s}_1),b}(\bar{m})$ is an elementary outer-extension of $tp_{f(\bar{s}_1)}(\bar{m})$.

Consider $tp_{f(\bar{s}_1)}(b)$. Since f is a partial elementary map there is some a satisfying the image under f^{-1} of $tp_{f(\bar{s}_1)}(b)$. We need now to show that the forth extension of f to a mapping of $\langle\bar{s}_1, a\rangle$ to $\langle\bar{s}_2, b\rangle$ will map a to b, which means that b is the first element of $\langle\bar{s}_2, b\rangle$ satisfying $tp_{f(\bar{s}_1)}(b)$. But this holds since otherwise some element of $\bar{m}$ would satisfy $tp_{f(\bar{s}_1)}(b)$ and there would be some partial elementary map fixing $f(\bar{s}_1)$ and sending $\bar{m}$ to a tuple $\bar{b}$ containing b. This tuple $\bar{b}$ would hence satisfy $tp_{f(\bar{s}_1)}(\bar{m})$ contradicting the fact that $tp_{f(\bar{s}_1),b}(\bar{m})$ is an elementary outer-extension of $tp_{f(\bar{s}_1)}(\bar{m})$. ⊣

We are now ready for the first half of our main theorem.

Lemma 5.7. *Let $\mathcal{M}$ be homogeneous structure. If $\mathcal{HE}(\infty) \neq \varnothing$ then Forth does not suffice.*

Proof. Our objective is to build two enumerations S_1, S_2 of M for which Forth is not onto. We must ensure that these two infinite sequences are really enumerations of M and don't miss any element of M. To this end, we start from two fixed enumerations S_1' and S_2' and pick elements from S_1' to build S_1 and elements from S_2' to build S_2. We can start with any enumerations, so S_1' and S_2' could be the same enumeration.

Initially the two enumerations S_1, S_2 are empty. We will alternatively pick the next element of S_1' not already in S_1 and add it to S_1 (possibly with some others elements of S_1' not already in S_1) and pick the next element of S_2' not already in S_2 and add it to S_2 (again possibly with some others elements of S_2' not already in S_2). This back-and-forth procedure will ensure that all elements of S_1' and S_2' have been picked and that therefore S_1 and S_2 both contain all elements of M.

More precisely, we will construct S_1 and S_2 by building, in an incremental fashion, two finite sequences $\bar{s}_1$ and $\bar{s}_2$ of distinct elements and a map f between them. At first f and both sequences $\bar{s}_1$ and $\bar{s}_2$ are empty. Furthermore we will always add new elements at the end of $\bar{s}_1$ and $\bar{s}_2$ and this infinite process will yield the respective enumerations S_1 and S_2. We will also always ensure the following invariant.

1. The map f is a Forth partial map with domain $\bar{s}_1$ and codomain $\bar{s}_2$ (i.e., the image of f is included in $\bar{s}_2$).

The first step of our construction is the following. Take some $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\infty)$. Note that since $\bar{a}$ and $\bar{m}$ have no common element, we have that no element of $\bar{a}$ satisfies $tp_{\bar{a}}(\bar{m})$. We first set $\bar{s}_1$ and $\bar{s}_2$ to be $\bar{a}$ and $\langle \bar{a}, \bar{m} \rangle$ respectively. Since $\bar{m}$ and $\bar{a}$ are disjoint, $\bar{s}_2$ is well defined (it does not contain duplicate elements). Furthermore the forth map f from S_1 to S_2 will fix $\bar{a}$ pointwise: $f(a) = a$ for all $a \in \bar{a}$, therefore invariant 1 holds at this point.

We will complete the enumerations S_1 and S_2 in such a way as to make the forth map avoid $\bar{m}$ (none of its elements will ever be in $Im(f)$), fulfilling our claim that f is not onto. In order to carry on the construction, we will furthermore assure the following invariants.

2. $tp_{f(\bar{s}_1)}(\bar{m}) \in \mathcal{HE}(\infty)$,
3. $m \notin f(\bar{s}_1)$, for all $m \in \bar{m}$,
4. $\bar{s}_2 = f(\bar{s}_1) \cup \bar{m}$ (as sets).

Since at this step of the construction $\bar{s}_1 = \bar{a}$, $\bar{s}_2 = \langle \bar{a}, \bar{m} \rangle$ and f is the identity on $\bar{a}$, all previous invariants are fulfilled.

Note that since f is a partial elementary map, it follows from invariant 2 above and Corollary 4.4 that the rank of the image of $tp_{f(\bar{s}_1)}(\bar{m})$ under f^{-1} is also infinite.

We now extend $\bar{s}_1$ and $\bar{s}_2$ by alternatively adding elements to them.

When it is $\bar{s}_1$'s turn to get a new element, we pick the next element s_1 of S_1', which is not already in $\bar{s}_1$.

The image $f(tp_{\bar{s}_1}(s_1))$ of $tp_{\bar{s}_1}(s_1)$ is a type realised in $M \backslash f(\bar{s}_1)$ by Proposition 2.1. Take an ordinal α such that $\mathcal{HE}(\infty) = \mathcal{HE}(\alpha) = \mathcal{HE}(\alpha + 1)$. Since $tp_{f(\bar{s}_1)}(\bar{m}) \in \mathcal{HE}(\infty)$ applying Proposition 5.5 with this α, one gets that there exists some $\bar{b}$, satisfying $f(tp_{\bar{s}_1}(s_1))$ and such that $tp_{f(\bar{s}_1),\bar{b}}(\bar{m}) \in \mathcal{HE}(\alpha) = \mathcal{HE}(\infty)$ and $tp_{f(\bar{s}_1),\bar{b}}(\bar{m})$ is an outer-extension of $tp_{f(\bar{s}_1)}(\bar{m})$. Since $tp_{f(\bar{s}_1),\bar{b}}(\bar{m})$ is an outer-extension of $tp_{f(\bar{s}_1)}(\bar{m})$, it follows that $\bar{b}$ is disjoint from both $f(\bar{s}_1)$ and $\bar{m}$. Hence $\bar{b}$ is disjoint from $\bar{s}_2$.

By Proposition 5.6, there exists some $\bar{a}'$ disjoint from $\bar{s}_1$ such that Forth extends f to a map from $\langle \bar{s}_1, \bar{a}' \rangle$ to $\langle \bar{s}_2, \bar{b} \rangle$ by sending $\bar{a}'$ onto $\bar{b}$. Since $\bar{a}'$ realises $tp_{\bar{s}_1}(s_1)$, we can assume without loss of generality that $s_1 \in \bar{a}'$. Add now $\bar{a}'$ and $\bar{b}$ at the ends of $\bar{s}_1$ and $\bar{s}_2$ respectively. Note that all of the above invariants hold.

Conversely, at $\bar{s}_2$'s turn, pick the next element s_2 of S_2', which is not already in $\bar{s}_2$.

Since $tp_{f(\bar{s}_1)}(\bar{m}) \in \mathcal{HE}(\infty)$, it follows from the fact that $\mathcal{HE}(\infty) = \mathcal{HE}(\mathcal{HE}(\infty))$ and the definition of T-expandable type, that either $tp_{f(\bar{s}_1)}(\bar{m})$ outer-extends beyond s_2 to a type in $\mathcal{HE}(\infty)$ or $tp_{f(\bar{s}_1)}(\bar{m}, s_2)$ outer-extends beyond s_2's type to a type in $\mathcal{HE}(\infty)$. This means that one of the following conditions holds.

1. there is some $\bar{b}$ containing s_2 such that $tp_{f(\bar{s}_1),\bar{b}}(\bar{m}) \in \mathcal{HE}(\infty)$ is an outer-extension of $tp_{f(\bar{s}_1)}(\bar{m})$.
2. there is some $\bar{b}'$ satisfying $tp_{f(\bar{s}_1)}(s_2)$ such that $tp_{f(\bar{s}_1),\bar{b}'}(\bar{m}, s_2) \in \mathcal{HE}(\infty)$ is an outer-extension of $tp_{f(\bar{s}_1)}(\bar{m}, s_2)$.

Note that it follows from the definition of outer-extension that in the first case $\bar{b}$ is disjoint from $\bar{s}_2 = \langle f(s_1), \bar{m}\rangle$ and that in the second case $\bar{b}'$ is disjoint from $\langle \bar{s}_2, s_2\rangle = \langle f(s_1), \bar{m}, s_2\rangle$.

If the first of these two conditions holds, one can proceed as previously applying Proposition 5.6 to obtain some $\bar{a}'$ disjoint from $\bar{s}_1$ such that Forth extends f to a map from $\langle \bar{s}_1, \bar{a}'\rangle$ to $\langle \bar{s}_2, \bar{b}\rangle$ by sending $\bar{a}'$ onto $\bar{b}$. We can therefore again add $\bar{a}'$ and $\bar{b}$ at the ends of $\bar{s}_1$ and $\bar{s}_2$ respectively, fulfilling all of the above invariants.

In the second case, we still apply Proposition 5.6 but this time considering the outer-extension $tp_{f(\bar{s}_1),\bar{b}'}(\bar{m}, s_2)$ of $tp_{f(\bar{s}_1)}(\bar{m}, s_2)$. This means that we are indeed updating $\bar{m}$ by adding s_2 at its end. Proposition 5.6 yields some $\bar{a}'$ disjoint from $\bar{s}_1$ such that Forth extends f to a map from $\langle \bar{s}_1, \bar{a}'\rangle$ to $\langle \bar{s}_2, \bar{b}'\rangle$ by sending $\bar{a}'$ onto $\bar{b}'$. Again we add $\bar{a}'$ and $\langle \bar{b}', s_2\rangle$ at the ends of $\bar{s}_1$ and $\bar{s}_2$ respectively. Invariant 1 and 2 clearly hold. Invariant 3 now states that $m \notin f(\bar{s}_1)$, for all m in the new value of $\bar{m}$, which now includes s_2. This last condition holds since $tp_{f(\bar{s}_1),\bar{b}'}(\bar{m}, s_2)$ is an outer-extension of $tp_{f(\bar{s}_1)}(\bar{m}, s_2)$. Finally Condition 4 also holds since $\bar{m}$ now contains s_2. This completes the proof. ⊣

§6. Ordinal rank: Forth suffices. We will prove in this section the final part of our main theorem, namely that if all types have ordinal rank, then Forth suffices.

Let us first note that the (ordinal) rank of a type is determined by the ranks of its outer-extensions, more precisely we have the following simple result.

PROPOSITION 6.1. *Let $tp_{\bar{a}}(\bar{m})$ be of rank α, an ordinal. There exists $b \in M$ such that both of the following conditions hold.*

1. *The rank of any outer-extension $tp_{\bar{a},\bar{b}}(\bar{m})$ of $tp_{\bar{a}}(\bar{m})$ such that $\bar{b}$ contains b, is strictly inferior to α.*
2. *The rank of any outer-extension $tp_{\bar{a},\bar{b}'}(\bar{m}, b)$ of $tp_{\bar{a}}(\bar{m}, b)$ such that $\bar{b}'$ realises $tp_{\bar{a}}(b)$, is strictly inferior to α.*

   In particular if $\alpha = 0$ there are no such outer-extensions.

PROOF. By the definition of the rank function $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\alpha)\backslash\mathcal{HE}(\alpha+1)$. By definition of the operator $\mathcal{HE}$, $\mathcal{HE}(\alpha+1) = \{p \in \mathcal{HE}(\alpha); p \text{ is } \mathcal{HE}(\alpha)\text{-expandable}\}$. Now if the statement of the proposition didn't hold, $tp_{\bar{a}}(\bar{m})$ would be in the set $\mathcal{HE}(\alpha+1)$, contradicting the hypothesis. ⊣

We are now ready to complete the proof of our main theorem.

LEMMA 6.2. *If $\mathcal{HE}(\infty) = \varnothing$ then Forth suffices.*

PROOF. Consider two enumerations $S_1 = \langle s_1, s_2, \dots\rangle$ and S_2 of M. We have to show that the corresponding Forth map f is onto.

We will consider, for k a natural number, the sets $F_k = \{f(s_1), \dots, f(s_k)\}$. Furthermore, for $\bar{m} \in M$, we will denote by $S_2(\bar{m})$ the set of all $f(s)$ $(s \in S_1)$ that appear before some element of $\bar{m}$ in the enumeration S_2.

Let us first establish the following fact. Let i, j be natural numbers and $\bar{m} \in M$ be such that $i < j$, $S_2(\bar{m}) \subseteq F_i$, and $\bar{m} \in M\backslash F_j$. Note that $S_2(\bar{m}) \subseteq F_i$ implies that the first element in the enumeration S_2 satisfying $tp_{f(s_1),\dots,f(s_i)}(m)$, for some $m \in \bar{m}$, is m itself. We claim that $tp_{f(s_1),\dots,f(s_k)}(\bar{m})$ is an outer-extension of $tp_{f(s_1),\dots,f(s_i)}(\bar{m})$, for all k satisfying $i < k \leqslant j$.

In order to establish this claim, it suffices to show that $tp_{f(s_1),\dots,f(s_k)}(\bar{m})$ is an elementary outer-extension of $tp_{f(s_1),\dots,f(s_{k-1})}(\bar{m})$, for k such that $i < k \leqslant j$. This holds since if $f(s_k)$ was in a tuple satisfying $tp_{f(s_1),\dots,f(s_{k-1})}(\bar{m})$ the Forth map f would send s_k to a value in $\bar{m}$ contradicting the fact that none of the element of $\bar{m}$ are in F_j.

We can now show that f is onto. Consider X the set of all tuples $\bar{m} \in M \backslash Im(f)$. We have to show that X is the empty set. To show this, take a $tp_{f(s_1),\dots,f(s_i)}(\bar{m})$ of minimal rank α, among all $\bar{m} \in X$ and i such that $S_2(\bar{m}) \subseteq F_i$. We will show that we can find such a type of even smaller rank, establishing as claimed that the set X is empty.

Indeed take b to be the element given by Proposition 6.1. If $b \in Im(f)$, take j such that $f(s_j) = b$. From Proposition 6.1, we have that the rank of $tp_{f(s_1),\dots,f(s_j)}(\bar{m})$ is strictly smaller than α, concluding this case.

To complete the proof, consider now the case $b \notin Im(f)$. Without loss of generality, one can consider that $S_2(\bar{m}, b) \subseteq F_i$, since otherwise take j big enough for $S(\bar{m}, b) \subseteq F_j$ to hold. By the above fact $tp_{f(s_1),\dots,f(s_j)}(\bar{m})$ is an outer-extension of $tp_{f(s_1),\dots,f(s_i)}(\bar{m})$. Now by Proposition 5.4 the rank of $tp_{f(s_1),\dots,f(s_j)}(\bar{m})$ is smaller or equal to α, which makes $tp_{f(s_1),\dots,f(s_j)}(\bar{m})$ a candidate of minimal rank. We hence can consider that $S_2(\bar{m}, b) \subseteq F_i$.

Take now j big enough for $f(s_1), \dots, f(s_j)$ to contain an element satisfying $tp_{f(s_1),\dots,f(s_i)}(b)$ (by homogeneity there is some element s_l satisfying the image under f^{-1} of $tp_{f(s_1),\dots,f(s_i)}(b)$ and $f(s_l)$ will satisfy $tp_{f(s_1),\dots,f(s_i)}(b)$ by definition of the Forth map). Using the above fact again, we have that $tp_{f(s_1),\dots,f(s_j)}(\bar{m}, b)$ is an outer-extension of $tp_{f(s_1),\dots,f(s_i)}(\bar{m}, b)$. Proposition 6.1 hence yields that the rank of $tp_{f(s_1),\dots,f(s_j)}(\bar{m}, b)$ is strictly smaller than α, contradicting the minimality of α and completing the proof. ⊣

§7. McLeish's Rank. In [13] McLeish gives a sufficient condition for Forth to suffice in term of an ordinal rank. It follows from Theorem 4.1 that if McLeish's condition holds then all type realised in $\mathcal{M}$ have ordinal rank as defined in this paper. In this section, we will show this result more directly by establishing that McLeish's rank is derived from a greatest fixpoint and is always greater or equal to ours.

Let $\mathcal{M}$ be a homogeneous structure and let $m, m' \in M$. Following Cameron [2, Section 5.2], the type $tp_{\bar{a},\bar{b}}(m')$ is said to *dominate* the type $tp_{\bar{a}}(m)$ in $\mathcal{M}$ if $tp^{\mathcal{M}}_{\bar{a},\bar{b}}(-/m') \subseteq tp^{\mathcal{M}}_{\bar{a}}(-/m)$. When $tp^{\mathcal{M}}_{\bar{a},\bar{b}}(-/m') \subsetneq tp^{\mathcal{M}}_{\bar{a}}(-/m)$, one says that $tp_{\bar{a},\bar{b}}(m')$ *strictly dominates* $tp_{\bar{a}}(m)$ in $\mathcal{M}$.

Furthermore $tp_{\bar{a}}(m)$ is said to be *splittable* if it is strictly dominated by some $tp_{\bar{a},\bar{b}}(m')$ for some tuple $\bar{b}$. In that case one also says that $tp_{\bar{a}}(m)$ is *split* by $tp_{\bar{a},\bar{b}}(m')$.

Note that if $tp^{\mathcal{M}}_{\bar{a}}(-/m)$ is a singleton, then it is never strictly dominated.

McLeish [13] considers the ordinal rank, which we will denote by $rank_{\mathcal{ML}}$, given by the greatest fixpoint of the following monotone operator.

$$\mathcal{ML}(T) = \{tp_{\bar{a}}(m) \in T; tp_{\bar{a}}(m) \text{ is split by some } q \in T\}.$$

Technically McLeish considers all types in one variable, even types $tp_{\bar{a}}(m)$ with $m \in \bar{a}$. But such a type is satisfied only by m and is hence never dominated. We can therefore restrict ourselves to types in $\mathcal{P}_{\mathcal{M}}$. Note that McLeish considers only 1-types, i.e., types in a single variable.

McLeish shows that if all 1-types of $\mathcal{P}_{\mathcal{M}}$ have ordinal McLeish-rank, then Forth suffices. However he also shows that there are homogeneous structures for which Forth suffices, but who do have types of infinite McLeish-rank (see also Section 8.4).

In fact McLeish's and our rank are related by the following relation.

Proposition 7.1. *For $\mathcal{M}$ a homogeneous structure, we have that $rk(tp_{\bar{a}}(m)) \leqslant rk_{\mathcal{ML}}(tp_{\bar{a}}(m))$ for all $m \in M$.*

Proof. It suffices to show that $\mathcal{HE}(\alpha) \subseteq \mathcal{ML}(\alpha)$ (for 1-types) for all ordinals α, since it then follows by induction on ordinals that $rk(tp_{\bar{a}}(m)) \leqslant rk_{\mathcal{ML}}(tp_{\bar{a}}(m))$.

We show the claim by induction on α. For $\alpha = 0$ and α a limit ordinal the case is clear. Take now $\alpha = \gamma + 1$ and $tp_{\bar{a}}(m) \in \mathcal{HE}(\gamma + 1)$.

Applying Proposition 5.5 with $\bar{m}$ the one-tuple m and $b = m$, we have that there exists $\bar{b}' \in M \backslash \bar{a}$ such that $\bar{b}'$ realises $tp_{\bar{a}}(m)$ and $tp_{\bar{a},\bar{b}'}(m) \in \mathcal{HE}(\gamma)$ is an outer-extension of $tp_{\bar{a}}(m)$.

We have by induction hypothesis that $tp_{\bar{a},\bar{b}'}(m) \in \mathcal{ML}(\gamma)$. It remains to be shown that $tp_{\bar{a},\bar{b}'}(m)$ splits $tp_{\bar{a}}(m)$. But since $\bar{b}'$ realises $tp_{\bar{a}}(m)$, there is some $b' \in \bar{b}'$ satisfying $tp_{\bar{a}}(m)$. Now this b' cannot satisfy $tp_{\bar{a},\bar{b}'}(m)$ since this type is an outer extension of $tp_{\bar{a}}(m)$. ⊣

It now follows from the previous proposition that if every 1-type has ordinal McLeish-rank then every 1-type also has ordinal rank in our sense. Furthermore from Proposition 5.3 it then follows that all types have ordinal rank. Forth hence suffices by Theorem 4.1, yielding a direct reduction of McLeish's result to ours.

But this is not the strongest result of McLeish. Indeed, McLeish shows [13, Theorem 3.9] that Forth suffices if for any type $tp_{\bar{a}}(m)$ there is a $\bar{b}$ with the following property: for any $\bar{c}$, $tp_{\bar{a},\bar{b},\bar{c}}(m')$ has ordinal McLeish's rank, when $tp_{\bar{a},\bar{b},\bar{c}}(m')$ strictly dominates $tp_{\bar{a}}(m)$. Let us show that here again this conditions implies that $tp_{\bar{a}}(m)$ has ordinal rank in our sense.

Take a $\bar{b}$ as in the previous paragraph. If $tp_{\bar{a}}(m)$ had infinite rank in our sense, then using induction on the length of $\bar{b}$, Proposition 5.5 would yield a $tp_{\bar{a},\bar{d}}(m)$ of infinite rank with $\bar{d}$ containing a subsequence satisfying $tp_{\bar{a}}(\bar{b})$. If $tp_{\bar{a},\bar{d}}(m)$ doesn't split $tp_{\bar{a}}(m)$, one can again use Proposition 5.5 to outer-extend it beyond $tp_{\bar{a}}(m)$, making sure it now splits $tp_{\bar{a}}(m)$. So without loss of generality we can assume that $tp_{\bar{a},\bar{d}}(m)$ splits $tp_{\bar{a}}(m)$.

Reordering the elements of $\bar{d}$ obviously doesn't change the set of elements satisfying the type, hence nor its rank (it could change the fact that it is an outer-extension though, but we don't need this fact here). We therefore have that $tp_{\bar{a},\bar{d}}(m)$ is of the form $tp_{\bar{a},\bar{b}',\bar{c}}(m)$ with $\bar{b}'$ satisfying $tp_{\bar{a}}(\bar{b})$.

Now by homogeneity, there is an automorphism of $\mathcal{M}$ fixing $\bar{a}$ and sending $\bar{b}'$ to $\bar{b}$. This automorphism will send $tp_{\bar{a},\bar{b}',\bar{c}}(m)$ to a type $tp_{\bar{a},\bar{b},\bar{c}'}(m')$ of same (infinite) rank. Furthermore since this automorphism fixes $\bar{a}$, $tp_{\bar{a},\bar{b},\bar{c}'}(m')$ also splits $tp_{\bar{a}}(m)$. Now by McLeish's hypothesis such a type has ordinal McLeish-rank hence by

Proposition 7.1 also ordinal rank in our sense, a contradiction. The type $tp_{\bar{a}}(m)$ has hence indeed ordinal rank in our sense.

§8. Examples. We present in this section the computation of our rank for various homogeneous structures. Our objective is to show explicit computations of our rank, exhibit representative examples and also compare our rank with McLeish's.

8.1. A finite graph. Let us first show that our rank can be nonzero even on a finite structure. Consider the 5-cycle C_5, which is the graph on 5 vertices forming a pentagon.

Since C_5 has a single orbit on one element, there is a single type over the empty sequence. This type cannot be outer-extended, it is therefore of rank 0. If one fixes at least two elements of C_5, all elements are fixed. So any type over a tuple of at least two elements will be satisfied by at most one element and all realised types of this form are of rank 0 by Proposition 4.2.

Finally, if one fixes a single element a, there are two orbits (hence types), each containing two elements. These types are of rank 1. Let us first show that these types are of rank at least 1.

Indeed, consider such a type $tp_a(m)$. This type is realised by exactly two elements, say m and m'. The type $tp_a(m)$ clearly outer-extends beyond any of the two remaining elements, since they do not satisfy $tp_a(m)$. Furthermore this type also outer extends beyond m', since in this case one just has to outer-extend first by an element not satisfying $tp_a(m)$ and then by m'. Finally one can outer-extend $tp_a(m)$ beyond m's type by again first extending by an element not realising $tp_a(m)$ and then by m'.

That the rank of $tp_a(m)$ is 1 follows from Proposition 5.5, since any outer-extension beyond m's type will fix at least a second element and as already mentioned will yield a type of rank 0.

As for types in more than one variable, we have from Proposition 5.3 that any type over the empty sequence or over a sequence of at least two elements is of rank 0. Furthermore, for a type $tp_a(\bar{m})$ over a single element a, as soon as $\bar{m}$ contains at least two elements, it either contains two elements satisfying the same type over a, say p_a, or one element in both types over a. In the first case one cannot outer-extend beyond p_a since $\bar{m}$ already contains the two element satisfying p_a. In the second case one cannot outer-extend beyond any of these two types over a. We hence have that $tp_a(\bar{m})$ is of rank 0.

As this example shows, our rank could be of some interest to finite combinatorics. A detailed analysis among these lines should be of interest, but is beyond the scope of this paper.

8.2. Countable dense linear order without endpoints. Let us consider Cantor's original example, namely that of a countable dense linear order without endpoints. As we already said, Cantor proved that Forth suffices in this case.

In a countable dense linear order without endpoints, a type $tp_{\bar{a}}(\bar{m})$ is determined by how the elements of $\bar{m}$ compare to those of $\bar{a}$. Namely, the elements of $\bar{a}$ divide the linear order into finitely many intervals and the type is determined by which of these intervals contain the elements of $\bar{m}$.

It hence follows that $tp_{\bar{a},b}(-/\bar{m}) = tp_{\bar{a}}(-/\bar{m})$, for any EOE $tp_{\bar{a},b}(\bar{m})$ of $tp_{\bar{a}}(\bar{m})$. Therefore for any $m \in \bar{m}$, we have that any element satisfying $tp_{\bar{a}}(m)$, will also satisfy any outer-extension $tp_{\bar{a},\bar{b}}(\bar{m})$ of $tp_{\bar{a}}(\bar{m})$. There are hence no outer-extensions of $tp_{\bar{a}}(-/\bar{m})$ beyond $tp_{\bar{a}}(m)$. It follows that all types are of rank 0.

8.3. Cameron's bicoloured countable dense linear order. Cameron presented in [2, Section 5.2] the first example of an $\aleph_0$-categorical structure for which Forth does not suffice. This structure is a dense linear order without endpoints, with furthermore a distinguish subset P that is both dense and co-dense (i.e., its complement is also dense). P and its complement therefore form a two-colour colouring of the structure.

As in the previous example, a tuple $\bar{a}$ divides this structure in finitely many intervals, which we will call $\bar{a}$*-cells*.

Consider some b in this structure. This b satisfies $tp_{\bar{a}}(\bar{m})$ if and only if it is of the same colour as some element of $\bar{m}$ appearing in its $\bar{a}$-cell. It therefore follows that if some $\bar{a}$-cell contains elements of $\bar{m}$ of both colour, the type $tp_{\bar{a}}(\bar{m})$ cannot be outer-extended beyond $tp_{\bar{a}}(b)$, for any b of this cell. In this case we therefore have that $tp_{\bar{a}}(\bar{m})$ is of rank 0.

Let us now show that all remaining type are of infinite rank.

When a type $tp_{\bar{a}}(\bar{m})$ is such that every cell created by $\bar{a}$ contains elements of $\bar{m}$ of at most one colour, we will say that it has *single coloured cells*.

We hence have to show that any single coloured cells type, $tp_{\bar{a}}(\bar{m})$, is in $\mathcal{HE}(\alpha)$ for all α. By induction, the claim obviously holds for $\alpha = 0$ and limit ordinals. As for a successor ordinal, take $\alpha = \gamma + 1$. Let us first show that for any $b \in M\backslash(\bar{a} \cup \bar{m})$, $tp_{\bar{a}}(\bar{m})$ extends beyond b to a single coloured cells type, which by induction hypothesis will be in $\mathcal{HE}(\gamma)$.

To show this first claim, note that by the above remark, if b is of a colour different than that of the element of $\bar{m}$ that are in its $\bar{a}$-cell, then $tp_{\bar{a},b}(\bar{m})$ is an outer-extension of $tp_{\bar{a}}(\bar{m})$. Clearly $tp_{\bar{a},b}(\bar{m})$ is again a single coloured cells type.

Otherwise if b is of the same colour as the elements of $\bar{m}$ contained in its cell, take b_1, b_2 elements not of b's colour but such that b is in the interval $[b_1, b_2]$ and no element of $\bar{m}$ is contained in this interval. We then have, similarly as above, that $tp_{\bar{a},b_1,b_2,b}(\bar{m})$ is an outer-extension of $tp_{\bar{a}}(\bar{m})$, completing the proof of the first claim.

Secondly, we show that for any $m \in \bar{m}$, $tp_{\bar{a}}(\bar{m})$ extends beyond $tp_{\bar{a}}(m)$ to a single coloured cells type. Since by induction this type will again be in $\mathcal{HE}(\gamma)$, both claims together will yield that $tp_{\bar{a}}(\bar{m})$ is $\mathcal{HE}(\gamma)$-expandable and therefore $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\alpha)$.

As for the second claim, one can take any $b \notin \bar{m}$, in the same $\bar{a}$-cell as m and of the same colour. This b clearly satisfies $tp_{\bar{a}}(m)$. We can now proceed as previously, taking b_1, b_2 elements not of b's colour but such that b is in the interval $[b_1, b_2]$ and no element of $\bar{m}$ is contained in this interval. As before $tp_{\bar{a},b_1,b_2,b}(\bar{m})$ is an outer-extension of $tp_{\bar{a}}(\bar{m})$, completing the proof.

8.4. McLeish's $\mathcal{M}$. McLeish [13, Example 2.5] considered the following structure $\mathcal{M}$. He furthermore showed that $\mathcal{M}$ is $\aleph_0$-categorical and that Forth suffices for it, but that it has types of infinite McLeish-rank [13, p. 884]. We will show that, as expected from Theorem 4.1, all types are indeed of ordinal rank.

$\mathcal{M}$ is the structure $\langle M, \equiv, P \rangle$, where M is a countable set, $\equiv$ an equivalence relation on M with infinitely many classes and P is a unary relation. P defines an infinite co-infinite subset of M. The complement $M \backslash P$ of P is denoted by Q.

Furthermore, the relation $\equiv$ restricted to Q is equality on Q. Also each element of Q is $\equiv$-equivalent to infinitely many elements of P. Finally P is divided into infinitely many infinite equivalence classes and each of these classes is $\equiv$-equivalent to a (unique) element of Q.

In this structure a type $tp_{\bar{a}}(m)$ is determined by which of P, Q contains m and which elements of $\bar{a}$ are $\equiv$-equivalent to m, as can be verified by building a suitable automorphism. Note that this makes use of the fact that by a type we mean, as said before, an element of $\mathcal{P}_{\mathcal{M}}$, hence we necessarily have that $m \neq a$ for all $a \in \bar{a}$. This is completely in accordance with the fact that our rank is defined on $\mathcal{P}_{\mathcal{M}}$ and therefore only elements of this set are relevant to our analysis.

If some element a of $\bar{a}$ is in the same $\equiv$-class as m, then an element m' satisfying $tp_{\bar{a}}(m)$, will also satisfy $tp_{\bar{a},\bar{b}}(m)$, for any $\bar{b}$. This follows from the fact that $m \equiv m'$ since they are both $\equiv$-equivalent to some $a \in \bar{a}$, hence m and m' are therefore also $\equiv$-equivalent to exactly the same $b \in \bar{b}$.

In order to compute the value of our rank, we will classify types $tp_{\bar{a}}(\bar{m})$ as to whether elements of P and/or Q appears in $\bar{m}$ and whether some elements of $\bar{m}$ are $\equiv$-equivalent to some elements of $\bar{a}$.

1. Let us first show that if some element of $\bar{m}$ is $\equiv$-equivalent to an element of $\bar{a}$, then $rk(tp_{\bar{a}}(\bar{m})) = 0$.

   Indeed, take $m \in \bar{m}$ and $a \in \bar{a}$ with $m \equiv a$. Since there is a single element of Q in any equivalence class, it follows from Proposition 4.2, that if $m \in Q$, then $rk(tp_{\bar{a}}(\bar{m})) = 0$.

   If $m \in P$, we will now show that there is some element b such that $tp_{\bar{a}}(\bar{m})$ doesn't extend beyond $tp_{\bar{a}}(b)$. By Corollary 5.2 this is sufficient to conclude that $rk(tp_{\bar{a}}(\bar{m})) = 0$.

   Take simply b, to be any element $\equiv$-equivalent to m, in P but not in $\bar{a}$. There is such an element since there are infinitely many elements in P, $\equiv$-equivalent to m. Note that to extend beyond $tp_{\bar{a}}(b)$, $tp_{\bar{a}}(\bar{m})$ must extend beyond some b', $\equiv$-equivalent to m, in P but not in $\bar{a}$. But now, by the previous remark, we have that b' and m satisfies the same type over any tuple containing $\bar{a}$. There are therefore no such outer-extension and the type is of rank 0 as claimed.
2. We now consider the case where no element of $\bar{m}$ is in the same class as an element of $\bar{a}$ but there are $m_i, m_j \in \bar{m}$, $m_i \in P$ and $m_j \in Q$. We will show that the rank is also 0 in this case.

   We show in fact that $tp_{\bar{a}}(\bar{m})$ cannot extend beyond any $tp_{\bar{a}}(b)$, for b in a class different from the elements of $\bar{a}$. From Corollary 5.2, it follows that $rk(tp_{\bar{a}}(\bar{m})) = 0$. In fact, since all elements satisfying $tp_{\bar{a}}(b)$ are in a class different from the elements of $\bar{a}$, it is sufficient to show that $tp_{\bar{a}}(\bar{m})$ cannot extend beyond any b that is in a class different from the elements of $\bar{a}$.

   Consider any type $tp_{\bar{a},\bar{b}}(\bar{m})$ such that $\bar{b}$ contains some element b that is in a class different from the elements of $\bar{a}$. Take b_k to be the first element of $\bar{b}$ not in the class of any $\bar{a}$ (since b is a candidate, there is such a b_k). Now neither b_k nor any element of $\bar{m}$ can be in the same class as some $b_1, \ldots, b_{k-1}$, since they would be in the class of some $\bar{a}$. It follows that if $b_k \in P$, it satisfies $tp_{\bar{a},b_1,\ldots,b_{k-1}}(m_i)$ and $tp_{\bar{a},\bar{b}}(\bar{m})$ isn't an outer-extension of $tp_{\bar{a}}(\bar{m})$. Similarly, if $b_k \in Q$, it satisfies $tp_{\bar{a},b_1,\ldots,b_{k-1}}(m_j)$ and $tp_{\bar{a},\bar{b}}(\bar{m})$ isn't an outer-extension of $tp_{\bar{a}}(\bar{m})$.

3. We now consider the case where no element of $\bar{m}$ is in the same class as an element of $\bar{a}$ but all elements of $\bar{m}$ are in P. We show that in this case the rank of $tp_{\bar{a}}(\bar{m})$ is 1.

   Take some b not in $\bar{a}$.

   First note that if b is in Q, then $tp_{\bar{a},b}(\bar{m})$ is an outer-extension of $tp_{\bar{a}}(\bar{m})$ since all elements of $\bar{m}$ are in P.

   Secondly, consider a $b \in P$ not $\equiv$-equivalent to any element of $\bar{m}$. Take $q \in Q$, in the same class as b. By the previous argument $tp_{\bar{a},q}(\bar{m})$ is an outer-extension of $tp_{\bar{a}}(\bar{m})$. Furthermore, since $b \equiv q$, while no element of $\bar{m}$ is equivalent to q, it follows that $tp_{\bar{a},q,b}(\bar{m})$ is an outer-extension of $tp_{\bar{a},p}(\bar{m})$. We therefore have that $tp_{\bar{a},q,b}(\bar{m})$ is an outer-extension of $tp_{\bar{a}}(\bar{m})$.

   Finally, for $b \in P$ and also $\equiv$-equivalent to some element of $\bar{m}$, note that any element of P not in the same class as any element of $\bar{a}$ satisfies the same type over $\bar{a}$ as b. Take such an element p, which furthermore is not in the same class as any element of $\bar{m}$ (this is possible since there are infinitely many classes in P). Take finally $q \in Q$ in the same class as this p. As before, $tp_{\bar{a},q,p}(\bar{m}, b)$ is an outer-extension of $tp_{\bar{a}}(\bar{m}, b)$.

   These three steps show that the rank of $tp_{\bar{a}}(\bar{m})$ is at least 1. In order to show that this rank is equal to 1, it is sufficient to exhibit a b with the following properties: any outer extension of $tp_{\bar{a}}(\bar{m})$ beyond b is of rank 0 and furthermore there is no outer-extension of $tp_{\bar{a}}(\bar{m}, b)$ beyond b's type.

   But this is easy to fulfil. Take $b \in Q$ and in the same $\equiv$-equivalence class as some element of $\bar{m}$. An outer-extension beyond b, is a Case 1 type, hence of rank 0. Furthermore $tp_{\bar{a}}(\bar{m}, b)$ is a Case 2 type, hence of rank 0.

4. The last case is when no element of $\bar{m}$ is in the same class as an element of $\bar{a}$ and all elements of $\bar{m}$ are in Q. Inverting P and Q, the proof of Case 3 shows that in this case also the rank is 1.

8.5. $\aleph_0$-categorical structures of finite ranks. For a fixed natural number n one can, by a construction similar to McLeish's $\mathcal{M}$'s, exhibit an $\aleph_0$-categorical structure $\mathcal{M}_n$ with the property that there is a type of rank r if and only if $r < n$. We will prove in this section the following result.

PROPOSITION 8.1. *For any natural number n there is an $\aleph_0$-categorical structure $\mathcal{M}_n$ with the property that there is a type of rank r if and only if $r < n$.*

Indeed, considered the following structure $\mathcal{M}_n = \langle M, \equiv, P_1, \ldots, P_n \rangle$, where M is a countable set, $\equiv$ an equivalence relation on M and $P_1, \ldots, P_n$ are unary relations. Let furthermore the relations $P_1, \ldots, P_n$, define a partition of M into infinite subsets. Finally, let $\equiv$ be an equivalence relation on M whose equivalence classes have exactly one element in every $P_1, \ldots, P_n$.

Since any countable structure elementary equivalent to $\mathcal{M}_n$ is in fact isomorphic to it, $\mathcal{M}_n$ is clearly $\aleph_0$-categorical.

In $\mathcal{M}_n$ a type $tp_{\bar{a}}(m)$ is determined by which $P_1, \dots, P_n$, contains m and which elements of $\bar{a}$ (if any) are $\equiv$-equivalent to m, as can be verified by building an automorphism.

Let us consider the type $tp_{\bar{a}}(\bar{m})$ of some tuple $\bar{m}$. If $m \equiv a$ for some $a \in \bar{a}$ and $m \in \bar{m}$, then m is the unique element of the P_i containing m that is $\equiv$-equivalent to a. The element m is hence the unique element satisfying its type over $\bar{a}$ and this type has by Proposition 4.2 rank 0.

If $m \not\equiv a$ for all $a \in \bar{a}$ and $m \in \bar{m}$, we will show that $rk(tp_{\bar{a}}(\bar{m}))$ is equal to the number $p_{\bar{m}}$ of $P_1, \dots, P_n$ containing no element of $\bar{m}$. This will suffices to prove that all types of $\mathcal{M}_n$ are of rank strictly smaller than n and also that for any $i < n$ there is a type of rank i. In order to show that $rk(tp_{\bar{a}}(\bar{m})) = p_{\bar{m}}$, it in fact suffices to prove the following result.

PROPOSITION 8.2. *If $m \not\equiv a$ for all $a \in \bar{a}$ and $m \in \bar{m}$, then for any natural number i, the two following conditions are equivalent.*

1. $p_{\bar{m}} \geqslant i$.
2. $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(i)$.

PROOF. The proof goes by induction on i.

The case $i = 0$ follows directly from the definitions. Suppose by induction that the result already holds for i and let us show that it also holds for $i + 1$.

Take first $tp_{\bar{a}}(\bar{m})$ to be a type satisfying the first condition: $p_{\bar{m}} \geqslant i+1$. In order to show that $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(i+1)$, we have to show that for any $b \in M \backslash \bar{a}$ either $tp_{\bar{a}}(\bar{m})$ outer-extends beyond b to a type in $\mathcal{HE}(i)$ or $tp_{\bar{a}}(\bar{m}, b)$ outer-extends beyond b's type to a type in $\mathcal{HE}(i)$. We will consider two cases.

Consider first a b that does not satisfy $tp_{\bar{a}}(\bar{m})$. In this case $tp_{\bar{a},b}(\bar{m})$ is an outer-extension of $tp_{\bar{a}}(\bar{m})$. Furthermore since $p_{\bar{m}} \geqslant i + 1 \geqslant i$, it follows from induction hypothesis that $tp_{\bar{a},b}(\bar{m}) \in \mathcal{HE}(i)$, as required.

Consider secondly a b that does satisfy $tp_{\bar{a}}(\bar{m})$. In this case, by hypothesis, b is not $\equiv$-equivalent to any element of $\bar{a}$. Take hence b' to be in the same $P_1, \dots, P_n$ as b and $\equiv$-equivalent to none of $\bar{a}$ nor $\bar{m}$ nor b. It hence follows that b' and b satisfy the same type over $\bar{a}$.

Take finally b'' to be in some $P_1, \dots, P_n$ containing no element of $\bar{m}$ nor b, but such that $b'' \equiv b'$. There is such a b'' since $p_{\bar{m}} \geqslant i + 1 > 0$. Now $tp_{\bar{a},b''}(\bar{m}, b)$ is an outer-extension of $tp_{\bar{a}}(\bar{m}, b)$ since b'' is not in the same $P_1, \dots, P_n$ as any element of $\bar{m}, b$. Furthermore, $tp_{\bar{a},b'',b'}(\bar{m}, b)$ is an outer-extension of $tp_{\bar{a},b''}(\bar{m}, b)$ since $b'' \equiv b'$ but b'' in not in the same $\equiv$-equivalence class as any element of $\bar{m}, b$. It hence follows that $tp_{\bar{a},b'',b'}(\bar{m}, b)$ is also an outer-extension of $tp_{\bar{a}}(\bar{m}, b)$. Finally $p_{\bar{m},b} \geqslant i$ since $p_{\bar{m}} \geqslant i + 1$, hence by induction hypothesis $tp_{\bar{a},b'',b'}(\bar{m}, b) \in \mathcal{HE}(i)$, as required.

Conversely, let the second condition, $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(i+1)$, hold. Let us first show that there is some $P_1, \dots, P_n$ containing no element of $\bar{m}$. Indeed, take any $m \in \bar{m}$. Since $i + 1 > 0$, it follows that $rk(tp_{\bar{a}}(\bar{m})) > 0$ and by Corollary 5.2 there is some outer extension of $tp_{\bar{a}}(\bar{m})$ beyond m's type. Consider hence such an outer-extension $tp_{\bar{a},\bar{b}}(\bar{m})$ and take the first $b_i \in \bar{b}$ that is $\equiv$-equivalent to no element of $\bar{a}$. There is such a b_i since the element of $\bar{b}$ satisfying the type of m is already a candidate (otherwise the rank would be 0 as we said before). Now since $tp_{\bar{a},b_1,\dots,b_i}(\bar{m})$ is an outer-extension of $tp_{\bar{a},b_1,\ldots,b_{i-1}}(\bar{m})$, there is no element of $\bar{m}$ in the same $P_1, \ldots, P_n$ as b_i. This element of $P_1, \ldots, P_n$ contains hence no element of $\bar{m}$ as required.

Take hence b to be in some $P_1, \ldots, P_n$ containing no element of $\bar{m}$ but such that $b \equiv m$ for some $m \in \bar{m}$. Since $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(i+1)$, either $tp_{\bar{a}}(\bar{m})$ outer-extends beyond b to a type in $\mathcal{HE}(i)$ or $tp_{\bar{a}}(\bar{m}, b)$ outer-extends beyond b's type to a type in $\mathcal{HE}(i)$.

Let us first consider the case where $tp_{\bar{a}}(\bar{m})$ outer-extends beyond b to a type in $\mathcal{HE}(i)$. In this case, since the rank of $tp_{\bar{a},\bar{b}}(\bar{m})$ is equal to 0 for any $\bar{b}$ containing b, it follows that $i = 0$. Now since there is some $P_1, \ldots, P_n$ containing no element of $\bar{m}$, $p_{\bar{m}} \geqslant 1 = i + 1$ and the first condition is satisfied.

If $tp_{\bar{a}}(\bar{m}, b)$ outer-extends beyond b's type to a type in $\mathcal{HE}(i)$, then by induction hypothesis $p_{\bar{m},b} \geqslant i$. Note also that by definition of b, $p_{\bar{m},b} = p_{\bar{m}} - 1$, hence $p_{\bar{m}} \geqslant i + 1$ as required. ⊣

8.6. Coloured wreath power of the integers. For α a countable ordinal, it remains to exhibit an homogeneous structure that has the property that all types are of rank strictly smaller than α and furthermore such that for any $\beta < \alpha$ there is a type of rank β. We will present in this section a proof of the following proposition.

PROPOSITION 8.3. *For any α a countable ordinal, there exists an homogeneous structure $Wr_{\omega,\alpha}\mathcal{Z}$ that has the property that the set of the ranks of its types is exactly α.*

Globally the construction method is based on the three following principles.

First, we consider a countably infinite structure whose automorphism group acts in a regular (sharply 1-transitive) way. We make use of the fact that this structure has a single type over the empty sequence, but that every element has a different type already over a single element.

Secondly, we use the *wreath power* construction of [6, Section 2] (see also [1, Section 6] and [13, Section 3.2]) to produce a structure parametrised by the ordinal α.

Unfortunately this is still not enough and we will finally consider an infinite disjoint union. Technically, this will allow to outer-extend types and have that the set of ranks is indeed α.

Let us now give an explicit construction. We first consider $\mathcal{Z} = \langle \mathbf{Z}, S \rangle$, where $\mathbf{Z}$ is the set of integers, while $S(x, y)$ is the successor relation holding when $x + 1 = y$. For any fixed value $k \in \mathbf{Z}$, the relation $R_k(x, y)$ holding for pairs (x, y) such that $x + k = y$ is definable in $\mathcal{Z}$. Note that in $\mathcal{Z}$, corresponding elements of two tuples satisfying the same formulas will have the same difference with (say) the first element (of its tuple). We also have that the automorphisms of $\mathcal{Z}$ are given by the mappings $x \mapsto x + k$, $k \in \mathbf{Z}$. The structure $\mathcal{Z}$ is hence homogeneous. Note also that there is a single type over the empty sequence. Finally as soon as the set of constants $\bar{a}$ is nonempty, every element has a different type over $\bar{a}$.

Let us now briefly present the wreath power construction. Consider α a countable ordinal and let S be some countable (finite or infinite) nonempty set. Fix some element $e \in S$ and consider functions $s : \alpha \to S$ (α-indexed sequences of elements of S) of *finite support*, which means that only finitely many values $s(\beta)$ are different from e. Let $S^{(\alpha)}$ be the set formed of all $s : \alpha \to S$ of finite support. This set is clearly countable.

For $\gamma < \alpha$ and $s, s' \in S^{(\alpha)}$, define $s \equiv_\gamma s'$ when $s(\beta) = s'(\beta)$ for all $\beta > \gamma$. Obviously $\equiv_\gamma$ is an equivalence relation. Note that for $s, s' \in S^{(\alpha)}$, $s \neq s'$, there is always a greatest ordinal γ such that $s(\gamma) \neq s'(\gamma)$. Furthermore, this ordinal is the smallest γ such that $s \equiv_\gamma s'$.

The structure $Wr_\alpha S = \langle S^{(\alpha)}, \equiv_\gamma; \gamma < \alpha \rangle$, introduced in [6, Section 2] (see also [1, Section 6] and [13, Section 3.2]), is called the *wreath power* of S (over α). This structure is homogeneous.

Let us introduce some terminology and notation. For $\gamma < \alpha$, we will say that a function $s : \{\beta \in \alpha; \beta > \gamma\} \to S$, is an γ-*tail*. Similarly, for s an element of $S^{(\alpha)}$, we will say that the restriction of s to $\{\beta \in \alpha; \beta > \gamma\}$ is *the* γ*-tail of* s.

The wreath power can be fruitfully seen as a tree whose leaves are the elements of $S^{(\alpha)}$, while inner nodes are γ-tails ($\gamma < \alpha$). A node n is a descendant of node n' if n' is the γ-tail of n for some $\gamma < \alpha$.

We apply the wreath power construction to our structure $\mathcal{Z}$. Namely consider $\mathbf{Z}^{(\alpha)}$ the set of finite support α-indexed sequences of integer, with 0 as distinguish element. Let also, for $\gamma < \alpha$, $S_\gamma(x, y)$ be the relation that holds when $x \equiv_\gamma y$ and $S(x(\gamma), y(\gamma))$. We consider the structure $Wr_\alpha \mathcal{Z} = \langle \mathbf{Z}^{(\alpha)}, \equiv_\gamma, S_\gamma; \gamma < \alpha \rangle$. Note that for any fixed value $k \in \mathbf{Z}$, the relation $R_{k,\gamma}(x, y)$ holding for pairs (x, y) such that $x \equiv_\gamma y$ and $x(\gamma) + k = y(\gamma)$ is definable in $Wr_\alpha \mathcal{Z}$.

As for the original wreath power, the structure $Wr_\alpha \mathcal{Z}$ can be shown to be homogeneous using the following family of automorphisms. Take f some automorphism of $\mathcal{Z}$, $\gamma < \alpha$ some ordinal and s a γ-tail. Define the function $f_{\gamma,s} : \mathbf{Z}^{(\alpha)} \to \mathbf{Z}^{(\alpha)}$ in the following way.

$$f_{\gamma,s}(s')(\beta') = \begin{cases} f(s'(\gamma)) & \text{when } \beta' = \gamma \text{ and } s \text{ is the } \gamma\text{-tail of } s', \\ s'(\beta') & \text{otherwise.} \end{cases}$$

Note that $f_{\gamma,s}$ modifies its argument at most on its γ-component, i.e., $f_{\gamma,s}(s')$ and s' are equal for all β's except maybe γ. Furthermore difference at γ occurs only when s is the γ-tail of s'. It hence follows that $f_{\gamma,s}$ preserves $\equiv_\beta$ for all $\beta < \alpha$. Furthermore since f is an automorphism of $\mathcal{Z}$ it follows that $f_{\gamma,s}$ preserves S_β for all $\beta < \alpha$ and $f_{\gamma,s}$ is indeed an automorphism of $Wr_\alpha \mathcal{Z}$.

In a way similar to the original wreath power construction, the structure $Wr_\alpha \mathcal{Z}$ is homogeneous, since two tuples of elements $\bar{s}, \bar{s}'$ of $\mathbf{Z}^{(\alpha)}$ of the same length satisfying the same formulas of the form $R_{k,\gamma}$ ($k \in \mathbf{Z}, \gamma < \alpha$) can be mapped to each other by some automorphism. This is usually referred to as homogeneity in the sense of Fraïssé (in the language $\{R_{k,\gamma}; k \in \mathbf{Z}, \gamma < \alpha\}$). To stay self-contained, we now prove this result.

Proposition 8.4. *Two tuples of elements* $\bar{s}, \bar{s}' \in \mathcal{Z}^{(\alpha)}$ *of the same length satisfy the same atomic formulas in the language* $\{R_{k,\gamma}; k \in \mathbf{Z}, \gamma < \alpha\}$ *if and only if these* $\bar{s}$, $\bar{s}'$ *can be mapped to each other by some automorphism of* $Wr_\alpha \mathcal{Z}$.

Proof. The right-to-left direction follows from the definition of automorphism.

For the left-to-right direction, assume that $\bar{s}, \bar{s}'$ satisfy the same atomic formulas in the language and consider an automorphism g of $Wr_\alpha \mathcal{Z}$. If g doesn't map $\bar{s} = (s_1, \dots, s_n)$ to $\bar{s}' = (s'_1, \dots, s'_n)$, there is a greatest i with $g(s_i) \neq s'_i$. One can then also consider the greatest ordinal β with $g(s_i)(\beta) \neq s'_i(\beta)$. For a contradiction, we will exhibit an automorphism with either a smaller i or for the same i a smaller β. This will show that there is indeed an automorphism mapping $\bar{s}$ to $\bar{s}'$.

First note that no $g(s_j) = s'_j$, for $j > i$, has the same β-tail as $g(s_i)$. Indeed, if $g(s_j) = s'_j$, for some $j > i$ had the same β-tail as $g(s_i)$, there would be some k with $R_{k,\beta}(g(s_i), g(s_j))$ hence $R_{k,\beta}(s_i, s_j)$ and also $R_{k,\beta}(s'_i, s'_j)$ would hold. But now, the fact that $g(s_j) = s'_j$ and therefore $g(s_j)(\beta) = s'_j(\beta)$ hold, would imply that $g(s_i)(\beta) = s'_i(\beta)$ contradicting the definition of β.

Take now f to be the automorphism of $\mathcal{Z}$ sending $g(s_i(\beta))$ to $s'_i(\beta)$. We claim that $h = f_{\beta,t} \circ g$, where t is the β-tail of $g(s_i)$, is the required automorphism. This is the case since $h(s_i)(\beta) = s'_i(\beta)$ follows from the definition of h, while $h(s_j) = s'_j$ for all $j > i$ follows from the fact no $g(s_j)$ $(j > i)$ has the same β-tail as $g(s_i)$. $\dashv$

The fact that $Wr_\alpha\mathcal{Z}$ is homogeneous in the sense of Fraïssé (in the language $\{R_{k,\gamma}; k \in \mathbf{Z}, \gamma < \alpha\}$) has as consequence that the type of a tuple is determined by the formulas $\{R_{k,\gamma}; k \in \mathbf{Z}, \gamma < \alpha\}$ that it satisfies.

Note first that if $\bar{a}$ is empty, all elements have the same type over $\bar{a}$ and this single type in one variable is of rank 0. Proposition 5.3 yields that types in any number of variables, over an empty sequence, are also of rank 0. Let us now consider nonempty sequences $\bar{a}$.

Consider $m, \bar{a} \in \mathcal{Z}^{(\alpha)}$. Take $r(m; \bar{a})$ to be the smallest ordinal γ such that $R_{k,\gamma}(m, a)$, for some $k \in \mathbf{Z}$, $\gamma < \alpha$ and $a \in \bar{a}$ (or equivalently such that $m \equiv_\gamma a$). Note that $R_{k',\gamma'}(m, a')$ if and only if either $\gamma' = \gamma$ and $R_{k'-k,\gamma'}(a, a')$ or $\gamma' > \gamma$ and $R_{k',\gamma'}(a, a')$.

We therefore have that an element m' satisfies the type of m over $\bar{a}$ if and only if the following conditions are satisfied (where $k \in \mathbf{Z}$ and $a \in \bar{a}$ are such that $R_{k,r(m;\bar{a})}(m, a)$).

1. $R_{k,r(m;\bar{a})}(m', a)$.
2. $\neg R_{k,\gamma}(m', a')$, for all $k \in \mathbf{Z}$, $\gamma < r(m; \bar{a})$ and all $a' \in \bar{a}$.

Note that in particular this means that any m' satisfying $tp_{\bar{a}}(m)$ must, like m, fulfil $R_{k,r(m;\bar{a})}(m', a)$. It hence follows that $m'(r(m; \bar{a})) = m(r(m; \bar{a}))$. This means that m' satisfies the type of m over $\bar{a}$ if and only if $m'(\gamma) = m(\gamma)$, for all $\gamma \geqslant r(m; \bar{a})$.

Unfortunately, we are still not done. In order to get a structure whose set of ranks is α, it is necessary to consider the disjoint union of ω copies of the previous structure. More precisely, we consider the set $\mathbf{Z}^{\omega,(\alpha)} = \omega \times \mathbf{Z}^{(\alpha)}$ of pairs (i, s) where i is intended to identify a copy of $\mathbf{Z}^{(\alpha)}$. For $(i, s), (i', s') \in \mathbf{Z}^{\omega,(\alpha)}$, we define $(i, s) \equiv_\gamma (i', s')$ if $s \equiv_\gamma s'$ and similarly $S_\gamma((i, s), (i', s'))$ if $S_\gamma(s, s')$. Finally we introduce unary predicates P_i $(i \in \omega)$ distinguishing the copies, i.e., $P_i((i', s'))$ holds exactly when $i = i'$. To simplify notation, for $u = (i, s) \in \mathbf{Z}^{\omega,(\alpha)}$, we will usually simply write $u(\gamma)$ for $s(\gamma)$.

Our final structure is hence $Wr_{\omega,\alpha}\mathcal{Z} = \langle \mathbf{Z}^{\omega,(\alpha)}; \equiv_\gamma, S_\gamma, P_i; \gamma < \alpha, i \in \omega\rangle$. Here again the relation $R_{k,\gamma}(x, y)$ holding for pairs $((i, x), (j, y))$ such that $x \equiv_\gamma y$ and $x(\gamma) + k = y(\gamma)$ is definable in $Wr_{\omega,\alpha}\mathcal{Z}$. We will show that the set of ranks of the types of this structure is equal to α, but before, let us show that this structure is homogeneous.

Note first that since equality between copies can be defined by $R_{0,0}(x, y)$, it follows that an automorphism of $Wr_{\omega,\alpha}\mathcal{Z}$ will move all copies jointly. In fact $Wr_{\omega,\alpha}\mathcal{Z}$ is homogeneous, since, as for $Wr_\alpha\mathcal{Z}$, two tuples of elements $\bar{s}, \bar{s}'$ of the same length satisfy the same atomic formulas in the language $\{P_i, R_{k,\gamma}; i \in \omega, k \in \mathbf{Z}, \gamma < \alpha\}$

exactly when they can be mapped to each other by some automorphism of $Wr_{\omega,\alpha}\mathcal{Z}$. This follows from the same property for $Wr_\alpha\mathcal{Z}$, as follows.

Consider $\bar{s}$, $\bar{s}'$ two tuples of elements of $Wr_{\omega,\alpha}\mathcal{Z}$ satisfying the same atomic formulas in the language $\{P_i, R_{k,\gamma}; i \in \omega, k \in \mathbf{Z}, \gamma < \alpha\}$. Let $\bar{s}_2$, $\bar{s}_2'$ be the projection of $\bar{s}$, $\bar{s}'$ (respectively) onto the second component $Wr_\alpha\mathcal{Z}$. The tuples $\bar{s}_2$, $\bar{s}_2'$ satisfy the same atomic formula (in $\{R_{k,\gamma}; k \in \mathbf{Z}\}$) in the structure $Wr_\alpha\mathcal{Z}$, there is hence an automorphism f of $Wr_\alpha\mathcal{Z}$ mapping $\bar{s}_2$ to $\bar{s}_2'$. This automorphism induces on $Wr_{\omega,\alpha}\mathcal{Z}$, by the mapping $(i, s) \mapsto (i, f(s))$, an automorphism sending $\bar{s}$ to $\bar{s}'$.

Let us now turn our attention to computing the ranks of the types of $Wr_{\omega,\alpha}\mathcal{Z}$. Define $r(\bar{m}; \bar{a})$ to be the minimum of the $r(m; \bar{a})$, $m \in \bar{m}$. We will now show that $rk(tp_{\bar{a}}(\bar{m})) = r(\bar{m}; \bar{a})$. This will show that α is indeed the set of ranks of the types of $Wr_{\omega,\alpha}\mathcal{Z}$. To show that $rk(tp_{\bar{a}}(\bar{m})) = r(\bar{m}; \bar{a})$, it will be sufficient to prove the following result.

Proposition 8.5. *For $m, \bar{a}$ elements of the structure $Wr_{\omega,\alpha}\mathcal{Z}$ the following conditions are equivalent.*

1. $r(\bar{m}; \bar{a}) \geqslant \beta$.
2. $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\beta)$.

Proof. We will show the result by ordinal induction.

The case $\beta = 0$ follows directly from the definitions.

As for a limit ordinal γ, note that $r(\bar{m}; \bar{a}) \geqslant \gamma$ exactly when $r(\bar{m}; \bar{a}) \geqslant \beta$ for all $\beta < \gamma$. From the definition, $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\gamma)$ exactly when $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\beta)$ for all $\beta < \gamma$. The equivalence at γ now follows from the induction hypothesis.

It remains to be shown that the equivalence holds at a successor ordinal $\beta + 1$.

For the first part of the equivalence, let us first suppose that $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\beta + 1)$. Take some $m \in \bar{m}$ such that $r(m; \bar{a}) = r(\bar{m}; \bar{a})$. Since $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\beta + 1)$ it follows from Proposition 5.5 with m as b that there is some $\bar{b} = b_1, \dots, b_n$ realising $tp_{\bar{a}}(m)$ and such that $tp_{\bar{a},\bar{b}}(\bar{m}) \in \mathcal{HE}(\beta)$ is an outer extension of $tp_{\bar{a}}(\bar{m})$. We therefore have $r(\bar{m}; \bar{a}, \bar{b}) \geqslant \beta$ from induction hypothesis.

The tuple $\bar{b} = b_1, \dots, b_n$ realises $tp_{\bar{a}}(m)$, take hence some $b \in \bar{b}$ satisfying $tp_{\bar{a}}(m)$. Note that since b satisfies $tp_{\bar{a}}(m)$ it follows that $b(\gamma) = m(\gamma)$, for all $\gamma \geqslant r(m; \bar{a})$. We hence have that $r(\bar{m}; \bar{a}) = r(m; \bar{a}) > r(m; \bar{a}, \bar{b}) \geqslant r(\bar{m}; \bar{a}, \bar{b}) \geqslant \beta$ and therefore $r(\bar{m}; \bar{a}) \geqslant \beta + 1$.

Let us now consider the second part of the equivalence, which is to show that $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\beta + 1)$ holds when $r(\bar{m}; \bar{a}) \geqslant \beta + 1$. First we have by induction hypothesis that since $r(\bar{m}, \bar{a}) \geqslant \beta + 1 > \beta$ it follows that $tp_{\bar{a}}(\bar{m}) \in \mathcal{HE}(\beta)$. It remains to be shown that $tp_{\bar{a}}(\bar{m})$ is $\mathcal{HE}(\beta)$-expandable. This means that we must show that for all $b \in \mathbf{Z}^{\omega,(\alpha)} \backslash \bar{a}$ either $tp_{\bar{a}}(\bar{m})$ outer-extends beyond b to a type in $\mathcal{HE}(\beta)$ or $tp_{\bar{a}}(\bar{m}, b)$ outer-extends beyond b's type to a type in $\mathcal{HE}(\beta)$.

Take hence $b \in \mathbf{Z}^{\omega,(\alpha)} \backslash \bar{a}$, and let δ be the smallest ordinal such that $b \equiv_\delta m$ for some $m \in \bar{m}$. We will consider two cases.

First, consider the case where $\delta > \beta$. Take then some b_1 with second component equal to b's but first component (the copy) different from those of all of $\bar{m}$. In this case $tp_{\bar{a},b_1}(\bar{m})$ is an outer extension of $tp_{\bar{a}}(\bar{m})$, since b_1 is in a copy different from all $\bar{m}$. Furthermore $tp_{\bar{a},b_1,b}(\bar{m})$ is an outer extension of $tp_{\bar{a},b_1}(\bar{m})$, since b_1 distinguishes b from all of $\bar{m}$. Indeed, $R_{0,0}(b_1, b)$ holds, but $R_{0,0}(b_1, m)$, for some $m \in \bar{m}$, would imply that $\delta = 0$, but since $\delta > \beta \geqslant 0$, this is clearly impossible. Finally, as

$\delta > \beta$, $r(\bar{m}; \bar{a}, b_1, b) \geqslant \beta + 1 > \beta$ and by induction hypothesis $tp_{\bar{a},b_1,b}(m) \in \mathcal{HE}(\beta)$, completing this case.

Secondly, consider the case where $\delta \leqslant \beta$. In this case one cannot necessarily directly outer-extend beyond b since this could make $r(\bar{m}; \bar{a}, b)$ much too small (for instance if $b \in \bar{m}$). Since $\delta \leqslant \beta$, there is some $m \in \bar{m}$, such that $b(\gamma) = m(\gamma)$ for all $\gamma > \beta$. Since $r(m; \bar{a}) \geqslant r(\bar{m}; \bar{a}) \geqslant \beta + 1 > \delta$, we have that $b(\gamma) = m(\gamma)$, for all $\gamma \geqslant r(m; \bar{a})$ and b satisfies the type of m over $\bar{a}$. Since m and b satisfy the same type over $\bar{a}$, it also follows that $r(\bar{m}, b; \bar{a}) = r(\bar{m}; \bar{a})$.

Take now b' to be some element in the same copy as b and such that $b'(\gamma) = b(\gamma)$ for all $\gamma > \beta$ while $b'(\beta) \notin \{b(\beta), m(\beta); m \in \bar{m}\}$. It hence follows that $r(\bar{m}, b; \bar{a}, b') \geqslant \beta$. Since $b'(\gamma) = m(\gamma)$ for all $\gamma > \beta$, b' satisfies the type of m (hence of b) over $\bar{a}$.

Take finally some element b_1 equal to b' at all ordinals but in some copy containing no $\bar{m}, b$. It hence follows that $r(\bar{m}, b; \bar{a}, b_1, b') = r(\bar{m}, b; \bar{a}, b') \geqslant \beta$. Furthermore $tp_{\bar{a},b_1}(\bar{m}, b)$ is an outer-extension of $tp_{\bar{a}}(\bar{m}, b)$, since no element of $\bar{m}, b$ is in the same copy as b_1. Also, $tp_{\bar{a},b_1,b'}(\bar{m}, b)$ is an outer-extension of $tp_{\bar{a},b_1}(\bar{m}, b)$ since b_1 distinguishes b' from all of $\bar{m}, b$. Indeed, $R_{0,0}(b_1, b')$ holds, but neither $R_{0,0}(b_1, b)$ nor $R_{0,0}(b_1, m)$, for some $m \in \bar{m}$, can hold since b' (and hence b_1) differ from all $\bar{m}, b$ at position β.

Finally since $r(\bar{m}, b; \bar{a}, b_1, b') \geqslant \beta$, it follows from induction hypothesis that $tp_{\bar{a},b_1,b}(\bar{m}) \in \mathcal{HE}(\beta)$ as required. ⊣

To conclude, one can note that $Wr_{\omega,\alpha}\mathcal{Z}$ is never $\aleph_0$-categorical, even when α is some natural number, since there are infinitely many types over the empty sequence. Nevertheless, a more detailed analysis of the previous proof would show that for a natural number α, it would be possible to replace **Z** by a finite cyclic group and reduce the number of copies to finitely many, in order to produce an $\aleph_0$-categorical structure that has the property that the set of ranks of its types is equal to this finite α. Since we already gave, in the previous section, a much simpler example covering this case, we will not develop in that direction any further.

There remains finally the question whether there are $\aleph_0$-categorical structures with types of any ordinal rank. This question is clearly of interest, since $\aleph_0$-categorical structures are arguably the most studied homogeneous structures. But as the last example seems to show, the construction of a structure with ranks above the natural numbers can be somewhat complex. On the other hand, $\aleph_0$-categoricity could constrain possible ordinal ranks. In any case, progress on this question should shed some light on the combinatorial nature of our rank.

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DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITÉ DU QUÉBEC À MONTRÉAL
C.P. 8888, SUCC. CENTRE-VILLE
MONTRÉAL, QUÉBEC, H3C 3P8, CANADA
E-mail: villemaire.roger@uqam.ca
URL: http://intra.info.uqam.ca/personnels/Members/villemaire_r