Theories of abelian Groups and Modules preserved under Extensions

Roger Villemaire*
Département d'informatique,
Université du Québec à Montréal
C.P. 8888, succ. centre-ville,
Montréal (Québec), Canada H3C 3P8,
villemaire.roger@uqam.ca

Michel Hébert
Department of Mathematics,
The American University in Cairo,
113 Sharia Kasr el Aini, P.O.Box 2511,
11511 Cairo, Egypt.

August 20, 2006

Abstract

A theory T of modules is preserved under extensions if for any submodule A of a module B, B is a model of T as soon as A and B/A are. We give a syntactic characterization of theories of modules preserved under extensions for the case of regular rings and also for the case of complete theory of abelian groups. This answers a question of U. Felgner.

*This research was partly supported by the National Science and Engineering Council of Canada.

1 Introduction

In this paper a *theory* is any consistent set of (first-order, finitary) sentences, a *complete theory* being a theory which contains either φ or $\neg\varphi$ for every sentence φ. Equivalently, it is the set $Th(M)$ (called the *theory of* M) of all sentences true in some structure M.

We will say that a theory or a sentence is *preserved under* some algebraic operation if its class of models is closed under this operation. Syntactic characterizations of such theories have been intensively studied in model theory, under the name of *preservation theorems*. Perhaps the first preservation theorem was the celebrated G. Birkhoff's 1935 result, stating that a class of algebraic structures is preserved under substructures, direct products and quotients if and only if it can be *defined by equations*, i.e. it is the class of all models of a set of sentences of the form $\forall\bar{x}(\phi(\bar{x}))$, where $\phi(\bar{x})$ is a conjunction of atomic formulas.

It turned out to be more difficult to establish preservation theorems for each one of the operations "substructure", "quotient" and (especially) "direct product". In the second half of the 1950's, closure under substructures was characterized by Łoś and Tarski by *universal sentences* ([1] Theorem 3.2.2), and closure under quotients was shown to correspond to *positive sentences* by R. Lyndon ([1] Theorem 3.2.4).

The problem of the direct products was much harder, and is more conveniently considered together with the more general case of the so-called *reduced products* (see [1]). From the 40's to the 70's, Mostowski, McKinsey, Feferman, Vaught, Horn, Chang, Keisler, Weinstein, Galvin and Shelah have all contributed to the solution of the two problems. [1] is a good source to follow the details of this adventure, but we just recall what is needed here.

Definition The set of *Horn formulas* is the smallest set of formulas containing finite disjunctions of negations of atomic formulas with at most one atomic formula, which is closed under conjunction, universal and existential quantifiers.

Horn showed that a theory axiomatized by Horn sentences is preserved under direct products. The converse was proved later for universal-existential theories (which were proved to be precisely the theories preserved under unions of chains of embeddings). However Chang and Morel showed that there are (existential-universal) theories which are not Horn but are nevertheless preserved under direct products. Finally Weinstein [10] gave a (rather involved) syntactic characterization of theories preserved under direct products, and Keisler proved, under the continuum hypothesis, that the Horn sentences are precisely those preserved under reduced products. Galvin later showed how to get rid of the continuum hypothesis.

In 1976, U. Felgner [3] showed that a complete theory of abelian groups is preserved under direct products if and only if it is a Horn theory. This result was further generalized by the first author to every (not necessarily complete) theory of modules ([9]).

Note that any theory is preserved under arbitrary products if and only if it is preserved under binary products ([1] Theorem 6.3.14). Since direct sums and direct products of modules are elementarily equivalent ([6] Corollary 2.24), a theory of modules is preserved under direct products if and only if it is preserved under binary direct sums.

Now the notion of extension in module theory can be seen as generalizing the concept of binary direct sum: a module B is said to be an *extension of the module C by the module A* if $C \cong B/A$. The direct sum $A \oplus C$ is an extension of C by A since $(A \oplus C)/A \cong C$.

During his doctoral studies at Tübingen in the late 1980's, the first author was asked by Professor Felgner if there could be some natural characterization of the theories of abelian groups preserved under extensions. No satisfactory characterization was found at the time, but the work done then was the starting point of [9] which was realized during his postdoctoral studies at McGill university.

The main objective of this work is to answer this question. We give such a characterization for a complete theory of abelian groups, in terms of the values of its Szmielew invariants.

The reader must be warned that the concept of preservation under extensions as defined in this paper is not equivalent to the one normally encountered in model-theory (such as in [1] Exercise 3.2.1), where an extension of N is just a structure containing N as a substructure. While the usual notion is meaningful for any first-order language, ours is specific to modules.

In Section 2, we review some basic facts about modules and their model theory. We give a simple syntactic characterization of the theories of modules over a regular ring which are preserved under extensions, and indicate possible avenues for more general cases.

Section 3 contains our main result (Theorem 3.4). After reviewing some basic facts about the model theory of abelian groups, we give a complete proof of our characterization of complete theories preserved under extensions.

2 Modules

We recall some terminology and well-known facts in module theory and its model theory, which we will use throughout the paper. The reader is referred to [6] for (a lot) more on the subject.

A *(short) exact sequence* is a sequence of homomorphisms

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0 \tag{1}$$

such that α is an embedding, β is surjective, and $im(\alpha) = ker(\alpha)$

Note that B is an extension of C by A if only if there exists such a sequence (1).

We will need to combine exact sequences in order to build new ones. Consider two exact sequences

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$$

$$0 \to A' \overset{\alpha'}{\to} B' \overset{\beta'}{\to} C' \to 0$$

The combination of these two exact sequences by component-wise application is the following sequence:

$$0 \to A \oplus A' \overset{\alpha\oplus\alpha'}{\longrightarrow} B \oplus B' \overset{\beta\oplus\beta'}{\longrightarrow} C \oplus C' \to 0$$

where $(\alpha \oplus \alpha')(a, a') = (\alpha(a), \alpha'(a'))$, and similarly for $\beta \oplus \beta'$. It is left to the reader to check that this sequence is indeed exact.

R being a fixed ring with identity, the language of the theory of (right) R-modules is the (first-order, finitary) language containing the neutral element 0, the operation of addition + and, for every $r \in R$, a unary function symbol which we will also denote by r.

The so-called *positive-primitive formulas* will be important in our context. Those are the ones of the form $\exists \bar{y}(\psi(\bar{x}, \bar{y}))$, with ψ a conjunction of atomic formulas. In any R-module, such a sentence is equivalent to one of a simpler form, which we take as our definition here:

Definition A *positive-primitive formula*, for short a *pp-formula*, is a formula which is equivalent to one of the form:

$$\exists \bar{y} \bigwedge_k (\sum_{i=1}^{n} x_i r_{i,k} + \sum_{j=1}^{m} y_j s_{j,k} = 0)$$

with $r_{i,k}, s_{j,k} \in R$, and $\bar{y} = y_1 ... y_m$.

If $A \xrightarrow{\alpha} B$ is an embedding and $\varphi(\bar{x})$ a pp-formula, one has that if $A \models \varphi[\bar{a}]$ then $B \models \varphi[\bar{a}]$. The converse is not necessarily true, but when it is, it gives the following important concept.

Definition An embedding $A \xrightarrow{\alpha} B$ is said to be *pure* if for every pp-formula $\varphi(\bar{x})$, one has that $A \models \varphi[\bar{a}]$ if and only if $B \models \varphi[\bar{a}]$.

In this definition, and in many of the facts about pp-formulas, one can assume that φ has only one free variable.

Definition An exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is said to be *pure-exact* if α is pure.

A theory is *preserved under pure extensions* if for any pure submodule A of B, B is a model of T as soon as A and B/A are. Equivalently, when in every pure-exact sequence as in the definition, B is a model of T when A and C are.

If $\varphi(x)$ is a pp-formula and M a module, the set $\varphi(M) = \{m \in M; M \models \varphi[m]\}$ is an abelian subgroup of M.

Definition Let $\varphi(x)$ and $\psi(x)$ be pp-formulas in one variable and M be a module. The *Bauer-Monk invariant* $Inv(M, \varphi, \psi)$ is the cardinality of the quotient abelian group $\varphi(M)/(\varphi(M) \wedge \psi(M))$ if it is finite, and ∞ otherwise.

The fundamental theorem of the model theory of modules is that it admits *pp-elimination of quantifiers*: every sentence in the language of R-modules is equivalent to a boolean combination of sentences of the form $Inv(M, \varphi, \psi) < k$, where $Inv(M, \varphi, \psi)$ is a Bauer-Monk invariant and k a natural number ([6] Corollary 2.15). We will use mainly the following obvious consequence:

Theorem 2.1 ([6] Corollary 2.18) *Two modules are elementarily equivalent if and only if their Bauer-Monk invariants are equal.*

We will see in Section 3 that for abelian groups, one can use still simpler invariants.

Let us now consider theories of modules preserved under extensions. First note that every such theory must be preserved under products (see the Introduction). The syntactic characterization of the theories of structures preserved under products is rather involved, but for modules it takes a particularly simple form, as we will see in the next theorem.

Examples of theories preserved under products but not under extensions are easily found, even for modules over the ring of integers, i.e., the abelian groups (see next section). However, any such theory must be preserved under pure extensions:

Theorem 2.2 *Let T be a theory of R-modules over some ring R. The following conditions are equivalent:*

(a) T is preserved under pure extensions;

(b) T is a Horn theory;

(c) T is preserved under reduced products;

(d) T is preserved under products.

If T is complete, then those conditions are equivalent to

(e) every Bauer-Monk invariant of T is either infinite or equal to 1*.*

Proof Note that the Bauer-Monk invariants of a *theory* T make sense when T is complete. (d) $\Leftrightarrow$ (c) is the main result of [9], and (b) $\Leftrightarrow$ (c) is a classical theorem of model theory ([1] Proposition 6.2.5′). (a) $\Rightarrow$ (d) follows from the fact that the natural embedding $A \to A \oplus A$ is pure. (d) $\Rightarrow$ (a) is [6], Lemma 2.23, stating that if B is a pure extension of C by A, then $B \equiv A \oplus C$. Finally, (d) $\Leftrightarrow$ (e) is [6], Lemma 2.23 and Corollary 2.18. ∎

Note that the fact that the Bauer-Monk invariants are infinite or equal to 1 can be easily expressed as Horn sentences, so for complete theories the values of the invariants already give an axiomatization in terms of Horn sentences.

We deduce immediately:

Corollary 2.3 *Let R be a (von Neumann) regular ring and T be a theory of R-modules. The following conditions are equivalent:*

(a) T is preserved under extensions;

(b) T is a Horn theory;

(c) T is preserved under products.

If $T = Th(M)$ is complete, then those conditions are equivalent to:

(d) for all idempotent r of R, $ann_M(r)$ is either 0 or infinite.

Proof (c) $\Rightarrow$ (a) is clear from Theorem 2.2 since all extensions are pure when R is regular ([6] Theorem 16.A (iv)). (c) $\Leftrightarrow$ (d) follows from the theorem above and a result of Rothmaler, showing that the Bauer-Monk invariants for modules over a regular ring have the required simple form ([6], Corollary 16.18). ∎

Corollary 2.3 suggests a possible approach for the general case. Trying to identify the form of the sentences preserved under extensions, we know already that they are special Horn sentences. However, we know also a condition on the elements of the ring which will make all Horn sentences preserved under extensions, namely that for every element r of the ring, there exists s such that $rsr = r$ (since this is equivalent to being regular by [6] Theorem 16.A (iii)). Could we trace down the reason for this connection at the syntactic level? In the case of complete theories, a similar approach could be attempted with the Bauer-Monk invariants instead of the Horn sentences.

We now turn our attention to a special case, namely the complete theories of abelian groups.

3 Abelian groups

In this section we review some basic notions about abelian groups which we will later need. We follow as closely as possible the notation and terminology of the classical reference on abelian groups [4].

Definition Let p be a prime number. A *p-element* is an element whose order is a power of p.

Definition An abelian group is said to be *p-torsion free* or to have *no p-torsion* if it contains no p-element.

We will need to consider some specific abelian groups which we now describe. The trivial group 0 is the group containing only the neutral element 0. Q will denote as usual the group of rational numbers. Q_p is the subgroup of Q formed of all fractions $\frac{n}{m}$ such that p does not divides m. Z is the group of integers, while mZ is the subgroup of Z formed of all multiples of m. The cyclic group of order m will be denoted by $Z(m)$, and we will see its elements as the cosets $n + mZ$ of Z/mZ. Finally the Prüfer group $Z(p^\infty)$ is the group formed of all p^nth roots of unity for $n \in N$, where N is the set of natural numbers. Note that $Z(p^\infty)$ is also $Q^{(p)}/Z$ where $Q^{(p)}$ is the group of fractions of the form $\frac{m}{p^n}$ (not to be confused with Q_p), where $m \in Z$ and $n \in N$.

Proposition 3.1 *Any extension of two p-divisible abelian groups is p-divisible.*

Proof Let

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

and take $b \in B$. Since $\beta(b)$ is p-divisible it is equal to some $pc \in C$. Take a pre-image $b' \in B$ of c under β. Now b and pb' are both mapped to the same value in C, hence $b - pb' = a$ for some $a \in A$. a being p-divisible we have that $a = pa'$ and therefore $b = p(b' + a')$, showing that b is p-divisible. ∎

Proposition 3.2 *Any extension of two p-torsion-free abelian groups is p-torsion-free.*

Proof Let

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

and take $b \in B$. If $pb = 0$ then $\beta(pb) = p\beta(b) = 0$ and since C has no p-torsion we have that $b \in A$. Again by hypothesis an element of order p of A must be 0 hence $b = 0$ completing the proof. ∎

3.1 Model theory of abelian groups

Szmielew in [8] showed that the first-order theory of abelian groups is decidable by showing that every formula is equivalent to a boolean combination of pp-formulas with *core sentences*, a concept that we define below.

Szmielew also showed that every abelian group is elementarily equivalent to one of a set of groups of specific forms, which we will call the *Szmielew* groups, following [2]. We now state the results of Szmielew which we will use, again following the presentation of [2].

Let us first introduce the Szmielew invariants. We denote by dim_p the vector space dimension over the field with p elements if it is finite, and ∞ otherwise. Similarly $|nG|$ is the cardinality of the subgroup $\{ng; g \in G\}$ if it is finite, and ∞ otherwise. Following [4], $G[p]$ is the subgroup of the abelian group G containing all the elements of order p and $nG[p]$ is a shorthand for $(nG)[p]$. The Szmielew invariants are the following values, where p is a prime number and n a natural number.

$$\begin{aligned} U(p,n;G) &= dim_p p^n G[p]/p^{n+1}G[p] \\ Tf(p,n;G) &= dim_p p^n G/p^{n+1}G \\ D(p,n;G) &= dim_p p^n G[p] \\ Exp(n;G) &= |nG| \end{aligned}$$

In fact the second and third invariants of [2] are a bit different than ours. They consider instead the values $Tf(p;G) = lim_{n\to\infty} Tf(p,n;G)$ and $D(p;G) = lim_{n\to\infty} D(p,n;G)$. $Tf(p;G)$ is well defined since multiplication by p is an epimorphism of $p^nG/p^{n+1}G$ onto $p^{n+1}G/p^{n+2}G$, so $Tf(p,n;G)$ decreases as n grows. Similarly $D(p;G)$ is well defined since $p^{n+1}G[p] \subseteq p^nG[p]$, so $D(p,n;G)$ decreases as n grows. Our choice of invariants does not change the validity of the results given below, but our version is more convenient for our purpose.

A *core sentence* is just any statement asserting that a Szmielew invariant is smaller than some specific natural number.

As in the case of modules, we have the following fundamental theorem for the model theory of abelian groups:

Theorem 3.3 ([2] Theorem 2.1 and 2.6) *Two abelian groups are elementarily equivalent if and only if all their Szmielew invariants are equal.*

Finally Szmielew introduced the following kind of groups and showed that every abelian group is elementarily equivalent to one of them ([2] Theorem 2.9).

Definition A *Szmielew group* is an abelian group of the following form where $\alpha_{p,n}$, β_p and γ_p are finite or countably infinite, δ is either 0 or 1, and where p ranges over the prime numbers and n ranges over the natural numbers (here $A^{(\alpha)}$ is the direct sum of α many copies of A):

$$\bigoplus_{p,n} Z(p^n)^{(\alpha_{p,n})} \oplus \bigoplus_{p} Q_p^{(\beta_p)} \oplus \bigoplus_{p} Z(p^\infty)^{(\gamma_p)} \oplus Q^{(\delta)} \qquad (2)$$

3.2 Complete theories of abelian groups preserved under extensions

In this section we characterize the complete theories of abelian groups preserved under extensions.

As for the modules, if $T = Th(G)$ is a complete theory, we can write $Inv(p, n; T)$ instead of $Inv(p, n; G)$ for any Szmielew invariant.

Theorem 3.4 *A complete theory of abelian groups T is preserved under extensions if and only if all of the following conditions are satisfied:*

(a) if $T \neq Th(0)$, then $Exp(n; T) = \infty$, for all $n > 0$.

(b) every other Szmielew invariant is either 0 or ∞;

(c) $U(p, n; T) = 0$ for all primes p and all natural numbers n;

(d) for any given prime p, $Tf(p, n; T)$ does not depend on n;

(e) for any given prime p, $D(p, n; T)$ does not depend on n;

(f) for any given prime p, $Tf(p, n; T)$ and $D(p, n; T)$ are not both infinite;

In order to prove the theorem, we will need the following lemmas.

Lemma 3.5 *A complete theory of abelian groups T preserved under extensions is either the theory of the trivial abelian group 0 or satisfies $Exp(n; T) = \infty$, for all $n > 0$.*

Proof First note that the following sequence

$$0 \to Z(p^n) \to Z(p^{2n}) \to Z(p^n) \to 0 \tag{3}$$

where the embedding sends the coset $x + p^n Z$ to $p^n \cdot x + p^{2n} Z$, is exact.

Secondly, by Theorem 2.2, $Exp(n; T)$ is ∞ or 1. In the latter case, a model G of T is n-torsion and, from the structure of such groups (as direct sums of cyclic groups of order bounded by n), it is clear from equation (3) that G must be 0, since otherwise T would not be closed under extensions. ∎

Lemma 3.6 *If T is a complete theory of abelian groups preserved under extensions, then $U(p, n; T) = 0$ for all primes p and all natural numbers n.*

Proof Let p be a prime number. We will show that there is a model G of T which is a Szmielew group having no cyclic p-group in its decomposition (2). The result will then follow since this implies that $U(p,n;G)=0$.

Let G' be a Szmielew group which is a model of T. Suppose G' has a cyclic p-group in the decomposition (2), and let $Z(p^n)$ be such a summand of smallest order. We then have that $U(p,n-1;G')\neq 0$. We will now show how to build an extension G of G' by G' which is again a Szmielew group but which has no direct summand which is a cyclic p-group of order equal to p^n. Therefore $U(p,n-1;G)=0$ and this is a contradiction since T is complete.

In the decomposition of G', regroup all summands of the form $Z(p^n)$ in order to write $G'=\bigoplus Z(p^n)\oplus G''$ where G'' contains no $Z(p^n)$ summand.

By component-wise application of the exact sequence (3), we also have an exact sequence

$$0\to\bigoplus Z(p^n)\to\bigoplus Z(p^{2n})\to\bigoplus Z(p^n)\to 0 \tag{4}$$

Finally since $G''\oplus G''$ is an extension of G'' by itself we get that

$$0\to G''\to G''\oplus G''\to G''\to 0 \tag{5}$$

is again exact. Combining the exact sequences (4) and (5) component-wise, we get the exact sequence

$$0\to\bigoplus Z(p^n)\oplus G''\to\bigoplus Z(p^{2n})\oplus G''\oplus G''\to\bigoplus Z(p^n)\oplus G''\to 0 \tag{6}$$

Taking G to be $\bigoplus Z(p^{2n})\oplus G''\oplus G''$ completes the proof. ■

Lemma 3.7 *Let T be a complete theory of abelian groups. For any prime p, if $U(p,n;T)=0$ for every natural number n, then $Tf(p,n;T)=Tf(p,0;T)$ for every natural number n.*

Proof This follows from Lemma 1.6 of [2], which is proved by showing the exactness of the following sequence:

$$0\to p^nG[p]/p^{n+1}G[p]\to p^nG/p^{n+1}G\xrightarrow{\times p}p^{n+1}G/p^{n+2}G\to 0$$

■

Lemma 3.8 *Let T be a complete theory of abelian groups. For any prime p, if $U(p,n;T)=0$ for every natural number n, then $D(p,n;T)=D(p,0;T)$ for every natural number n.*

Proof This follows from Lemma 1.8 of [2], which is proved by showing the exactness of the following sequence:

$$0 \to p^{n+1}G[p] \to p^n G[p] \to p^n G[p]/p^{n+1}G[p] \to 0$$

$\blacksquare$

In the following lemmas we will use some standard homomorphisms between Q, Q_p, $Z(p^\infty)$ and $Z(p)$, in order to construct specific extensions. Hence it may be useful to recall some well-know related facts:

1. Q_p is a subgroup of the rationals Q, and the mapping $x \mapsto \frac{x}{n}$ where n is a non-zero integer is an embedding of Q_p into Q.
2. $Q_p/pQ_p \cong Z(p)$.
3. $Q/Q_p \cong Z(p^\infty)$.
4. The (division by p) mapping $(n+pZ) \mapsto (\frac{n}{p})$ is a well defined embedding of $Z(p)$ into $Z(p^\infty)$.
5. The (division by p) mapping $(\frac{n}{m} + pQ_p) \mapsto (\frac{n}{p \cdot m} + Q_p)$ is a well defined homomorphism from Q_p/pQ_p to Q/Q_p.

Lemma 3.9 *There is an exact sequence of the form*

$$0 \to Q_p \to Q \oplus Z(p) \to Z(p^\infty) \to 0 \tag{7}$$

Proof By the facts above, it is sufficient to show that the following sequence is exact:

$$0 \to Q_p \xrightarrow{\alpha} Q \oplus Q_p/pQ_p \xrightarrow{\beta} Q/Q_p \to 0 \tag{8}$$

where $\alpha(q) = (\frac{q}{p}, q + pQ_p)$ and $\beta(q, q' + pQ_p) = (q + Q_p) - (\frac{q'}{p} + Q_p)$

α is an embedding by the fact 1, and β is an epimorphism because the canonical homomorphism $Q \to Q/Q_p$ is onto.

$im(\alpha) \subseteq ker(\beta)$ is clear from $\beta(\alpha(q)) = \beta(\frac{q}{p}, q + pQ_p) = (\frac{q}{p} + Q_p) - (\frac{q}{p} + Q_p) = 0$.

In order to show that $ker(\beta) \subseteq im(\alpha)$, consider $(q, q' + pQ_p)$ such that $\beta(q, q' + pQ_p) = (q + Q_p) - (\frac{q'}{p} + Q_p) = 0$. This means that $q + Q_p = \frac{q'}{p} + Q_p$ and hence $q = \frac{q'}{p} + \frac{n}{m} = \frac{1}{p}(\frac{q' \cdot m + p \cdot n}{m})$, with $\frac{n}{m} \in Q_p$. Therefore $(q, q' + pQ_p) = (\frac{1}{p}(\frac{q' \cdot m + p \cdot n}{m}), (\frac{q' \cdot m + p \cdot n}{m}) + pQ_p) = \alpha(\frac{q' \cdot m + p \cdot n}{m})$, as required. $\blacksquare$

Lemma 3.10 *If T is a complete theory of abelian groups preserved under extensions, then for every prime p, $Tf(p,n;T)$ and $D(p,n;T)$ are not both infinite.*

Proof This proof is similar in spirit to the one of Lemma 3.6.

Assuming that for some prime p, $Tf(p,n;T) = D(p,n;T) = \infty$, we will build an extension G of two models of T such that $U(p,0;G) \neq 0$, contradicting Lemma 3.6.

So take G' to be a Szmielew group which is a model of T satisfying $Tf(p,n;T) = D(p,n;T) = \infty$ for some prime p. Since by Lemma 3.6 we have that $U(p,n;G') = 0$ for every natural number n, it follows that the decomposition of G' has at least one copy of Q_p and also one copy of $Z(p^\infty)$.

Hence $G' = Q_p \oplus G'' = Z(p^\infty) \oplus G'''$ for some G'' and G'''.

Since $G'' \oplus G'''$ is an extension of G'' by G''' we get the exact sequence

$$0 \to G'' \to G'' \oplus G''' \to G''' \to 0 \tag{9}$$

Combining the exact sequences (7) and (9) component-wise, we obtain the exact sequence

$$0 \to Q_p \oplus G'' \to Q \oplus Z(p) \oplus G'' \oplus G''' \to Z(p^\infty) \oplus G''' \to 0 \tag{10}$$

Taking G to be $Q \oplus Z(p) \oplus G'' \oplus G'''$ completes the proof. ∎

We can now give the first part of the proof of the theorem.

Proof of Theorem 3.4 (left to right) Condition (a) follows from Lemma 3.5. Condition (b) follows from the fact that if T is preserved under extensions, then it is preserved under direct products, therefore its invariants (other than $Exp(n;T)$) are either 0 or infinite.

The others conditions follow from Lemmas 3.6, 3.7, 3.8, and 3.10 respectively. ∎

In order to complete the proof of Theorem 3.4, we need some more lemmas.

Lemma 3.11 *Let T be a complete theory of abelian groups. If $Tf(p,n;T) = \infty$, then $Tf(p,n;B) = \infty$ for any extension B of two models of T.*

Proof Let A and C be two models of T in the exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

By hypothesis we have that $dim_p p^n C/p^{n+1}C = \infty$. Take infinitely many $c_i \in p^n C$ which are linearly independent modulo $p^{n+1}C$. Each c_i has a pre-image b_i under β which is in $p^n B$ (if $c_i = p^n c$ take p^n times a pre-image of c). These b_i's must be linearly independent modulo $p^{n+1}B$ since the c_i's are independent modulo $p^{n+1}C$. ∎

Lemma 3.12 *Let T be a complete theory of abelian groups. If $D(p, n; T) = \infty$, then $D(p, n; G) = \infty$ for any extension G of two models of T.*

Proof Let A and C be two models of T in the exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

By hypothesis we have that $dim_p p^n A[p] = \infty$. Now α sends elements of $p^n A[p]$ to element of $p^n B[p]$, proving the claim. ∎

Lemma 3.13 *Let T be a complete theory of abelian groups. If $Exp(n; T) = \infty$, then $Exp(n; G) = \infty$ for any extension G of two models of T.*

Proof Let A and C be two models of T in the exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

By hypothesis we have that $|nA| = \infty$. α sends elements of nA to elements of nB, proving the claim. ∎

Proof of Theorem 3.4 (right to left) Let T be a complete theory of abelian groups satisfying the conditions (a) to (f) of the theorem. We will show that T is preserved under extensions by showing that the value of every invariant is preserved under extensions, i.e. an extension of two models of T has the same invariants has T.

We first have to show that an extension B of two models A and C of T satisfies $U(p, n; B) = 0$ for all prime numbers p and all natural numbers n.

Consider the following exact sequence:

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \qquad (11)$$

Take $b \in p^n B$. We will show that $b \in p^{n+1}B[p]$, showing that $U(p,n;B) = 0$.

Now since β is a homomorphism, we have that $\beta(b) \in p^n C[p]$. Since $U(p,n;C) = 0$ it follows that $\beta(b) \in p^{n+1}C[p]$, which means that there is a $c \in C$ such that $p^{n+1}c = \beta(b)$. Take a pre-image b' of this c under β and let b'' be $p^{n+1}b'$. Since both b and b'' map to the same element under β we have $b = b'' + a$ for some $a \in A$. Now by hypothesis either $Tf(p,n;T) = 0$ or $D(p,n;T) = 0$.

In the first case $dim_p p^m A/p^{m+1}A = 0$ for every natural number m, and therefore every element of A is divisible by p^m for every m. Hence a is divisible by p^{n+1}, and since $b = b'' + a$ and b'' is also divisible by p^{n+1}, we have that b is also divisible by p^{n+1}. Therefore $b \in p^{n+1}B[p]$, proving the claim.

In the case where $D(p,n;T) = 0$, from $\beta(b) \in p^n C[p]$ it follows that $\beta(b) = 0$, and hence $b \in A$. Now b is an element of order p of A. Since $U(p,n;A) = 0$ for every natural number n, it follows by definition of $U(p,n;A)$ that $p^n A[p]/p^{n+1}A[p] = 0$ for all n. Therefore b is divisible in A (and hence also in B) by every power of p. It follows that $b \in p^{n+1}B[p]$ completing the proof of the claim.

Let us now consider the second invariant $Tf(p,n;T)$. By Lemma 3.11, if $Tf(p,n;T)$ is infinite then this also holds for any extension of two models of T. We will now show that if $Tf(p,n;T) = 0$, then $Tf(p,n;G) = 0$ for any extension G of two models of T.

By Lemma 3.7, $Tf(p,n;T) = Tf(p,0;T)$, so $Tf(p,0;T)$ is also equal to 0. Now $p^0 H/pH = H/pH$, so $Tf(p,0;H) = 1$ is equivalent to $H/pH = 0$. This means that H is p-divisible. Hence every model H of T is p-divisible. Now if G is an extension of two models of T, then by Proposition 3.1 G is p-divisible, so that $Tf(p,0;G) = 0$. We have already shown that $U(p,n;G) = 0$ for all n since it is an extension of two models of T, so by Lemma 3.7 it follows that $Tf(p,n;G) = 0$ for all n. This completes the proof of the preservation of $Tf(p,n;T)$ by extension.

For the third invariant $D(p,n;T)$, as in the last case it follows from Lemma 3.12 that if $D(p,n;T) = \infty$, then this also holds for any extension of two models of T. We now show that if $D(p,n;T) = 0$, then $D(p,n;G) = 0$ for any extension G of two models of T.

By Lemma 3.8, $D(p,n;T) = D(p,0;T)$, so $D(p,0;T)$ is also equal to 0. Now $D(p,0;H) = dim_p H[p]$, so $D(p,0;H) = 0$ is equivalent to $H[p] = 0$, which means that H has no p-torsion. We therefore have that every model

H of T has no p-torsion. Consider an extension G of two models of T. By Proposition 3.2, G has no p-torsion, so $D(p, 0; G) = 0$. We have already shown that $U(p, n; G) = 0$ for all n since it is an extension of two models of T, so by Lemma 3.8 it follows that $D(p, n; G) = 0$ for all n. This completes the proof of the preservation of $D(p, n; T)$ by extension.

The theory of the trivial abelian group 0 is obviously preserved under extensions. If T is not the theory of 0 and fulfill all conditions of the statement of the theorem, then $Exp(n; T) = \infty$ for all $n > 0$ by hypothesis. Now by Lemma 3.13 we have that $Exp(n; G) = \infty$ for any G which is an extension of two models of T. This completes the proof. ∎

Theorem 3.4 characterizes complete theories of abelian groups preserved under extensions in terms of their Szmielew invariants. Alternatively the following result, whose proof consists in computing the Szmielew invariants, characterizes these theories in terms of their Szmielew groups.

Corollary 3.14 *The theory $Th(G)$ of the abelian group G is preserved by extensions if and only if G is elementarily equivalent to a group of the following form:*

$$\bigoplus_{p \in P_1} Q_p^{(\omega)} \oplus \bigoplus_{p \in P_2} Z(p^\infty)^{(\omega)} \oplus Q^{(\delta)}$$

where P_1, P_2 are disjoint sets of prime numbers and δ is either 0 *or* 1*.*

4 Conclusion

We have characterized theories preserved under extensions for modules over regular rings, and for complete theories of abelian groups. Different types of characterizations appear in the paper: structural, syntactical, and in terms of the values of the (modules or groups) invariants.

Obvious open problems remain: 1) find a simple syntactically defined family of sentences which characterize theories of R-modules (or abelian groups) preserved under extensions, 2) formulate a more general characterization in terms of the (modules or abelian groups) invariants for the theories preserved under extensions.

A possible approach for 1) was mentioned in Section 2. Note however that there is no guarantee that preservation under extensions is a uniform property: this means that there might be a theory preserved under extensions which is not equivalent to any set of sentences which are themselves (individually) preserved under extensions. Such a possibility was first recognized by M. Rabin, who showed that preservation under intersection (of substructures) is not uniform in this sense ([7]). For more on this see [5].

As for 2), an idea could be to try to describe the possible invariants of an extension of two modules or abelian groups A and C in terms of the invariants of A and of C. Both suggestions appear to be rather difficult, but this problem surely deserves further study.

On a more personal level, the first author would like again to thank Professor Felgner for supervising him at the doctoral level, sharing his enthusiasm for research and logic. It is a pleasure to have the chance to return after so many years to the field of model theory, particularly by contributing to this longstanding open problem which was initiated by Professor Felgner.

References

[1] Chen Chung Chang and H. Jerome Keisler. *Model theory, 3rd ed.*, volume 73. North-Holland, 1990.

[2] Paul C. Eklof and Edward R. Fisher. The elementary theory of abelian groups. *Annals of Mathematical Logic*, 4(2):115–171, 1972.

[3] Ulrich Felgner. Horn theories of abelian groups. In *Model theory of algebra and arithmetic*, volume 834 of *Lecture Notes in Mathematics*, pages 169–173. Springer-Verlag, 1980.

[4] Laszlo Fuchs. *Infinite abelian groups*, volume 1-2. Academic Press, 1970-1973.

[5] Michel Hébert. Syntactic characterization of closure under connected limits. *Archive for Mathematical Logic*, 31(2):133–143, 1991.

[6] Mike Prest. *Model theory and Modules*, volume 130. Cambridge University Press, 1988.

[7] Michael O. Rabin. Classes of models and sets of sentences with the intersection property. *Annales scientifiques de l'université de Clermont*, 7:39–53, 1962.

[8] Wanda Szmielew. Elementary properties of abelian groups. *Fundamentae Mathematica*, 41:203–71, 1955.

[9] Roger Villemaire. Theories of modules closed under direct products. *Journal of Symbolic Logic*, 57(2):515–521, 1992.

[10] Joseph Michael Weinstein. *First-order properties preserved by direct product*. PhD thesis, University of Wisconsin, Madison, 1965.